

Errata

The Joy of Sets: Fundamentals of Contemporary Set Theory by Keith Devlin

Due to a production error, the symbol \upharpoonright for function restriction was omitted throughout the book. It occurs 70 times, on 21 pages, as follows:

p. 14, 4 times, bottom of page:

$$f \upharpoonright u = \{(a, f(a)) \mid a \in u\}.$$

Notice that $f \upharpoonright u$ is a function, with domain u .

Exercise 1.6.4. Prove that if $f : x \rightarrow y$ and $u \subseteq x$, then

$$(i) \ f[u] = \text{ran}(f \upharpoonright u);$$

$$(ii) \ f \upharpoonright u = f \cap (u \times \text{ran}(f)).$$

Let $f : x \rightarrow y$. We say f is *injective* (or *one-one*) if and only if

$$a \neq b \rightarrow f(a) \neq f(b).$$

p. 21, 1 time, Eq. (2):

$$(2) \quad (g_y \upharpoonright X_x) : X_x \cong (Z(y))_{g_y(x)}.$$

p. 25, 1 time, following first displayed equation:

denotes $f \upharpoonright \beta$. This clearly gives a precise meaning to what we generally

p. 51, 2 times, preceding Sect. 2.6:

Definitions of the above kind are sometimes referred to as definitions 'by induction'. More correctly they are definitions by *recursion*. (*Induction* is a method of proof, not of definition.) Letting $f : \text{On} \rightarrow V$ (use of class notation!) be the 'function' $f(\alpha) = V_\alpha$, we define $f(\alpha)$ in terms of $f \upharpoonright \alpha$ (i.e. in terms of $\langle f(\beta) \mid \beta < \alpha \rangle$). Indeed, we have

$$f(\alpha) = \bigcup \{ \mathcal{P}((f \upharpoonright \alpha)(\beta)) \mid \beta < \alpha \}.$$

That such definitions are possible in ZF set theory is a consequence of the recursion principle, which I consider next.

p. 52, 2 times, 1st and 3rd displayed equations:

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha).$$

$$\phi(\alpha, f \upharpoonright \alpha, f(\alpha)).$$

p. 53, 3 times, 3rd, 5th, and 6th displayed equations:

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha)$$

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha)$$

$$f_i(\alpha) = h(\alpha, f_i \upharpoonright \alpha).$$

p. 54, 13 times, middle of page:

Now assume $\mu > 0$ and that the result holds for all $\mu' < \mu$. Thus, for $\mu' < \mu$, $f_1 \upharpoonright \mu' = f_2 \upharpoonright \mu'$. If μ is a limit ordinal, then it follows at once that $f_1 = f_2$. Otherwise, let $\mu = \nu + 1$. Then we have, by the induction hypothesis, $f_1 \upharpoonright \nu = f_2 \upharpoonright \nu$. Hence

$$f_1(\nu) = h(\nu, f_1 \upharpoonright \nu) = h(\nu, f_2 \upharpoonright \nu) = f_2(\nu).$$

Thus,

$$f_1 = (f_1 \upharpoonright \nu) \cup \{(\nu, f_1(\nu))\} = (f_2 \upharpoonright \nu) \cup \{(\nu, f_2(\nu))\} = f_2,$$

which completes the proof. \square

Turning to the proof of the existence part of Theorem 2.6.1, let M be the class

$$M = \{f \mid (\exists \mu \leq \lambda)[(f : \mu \rightarrow V) \wedge (\forall \alpha \in \mu)(f(\alpha) = h(\alpha, f \upharpoonright \alpha))]\}.$$

In order to prove Theorem 2.6.1, it suffices to show that there is a function $f \in M$ such that $\text{dom}(f) = \lambda$.

Lemma 2.6.3 Let $f, g \in M$. Let $\mu = \text{dom}(f)$, $\nu = \text{dom}(g)$, and suppose $\mu < \nu$. Then $f = g \upharpoonright \mu$.

Proof: For all $\alpha \in \mu$, we have

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha),$$

$$g(\alpha) = h(\alpha, g \upharpoonright \alpha).$$

So, by Lemma 2.6.2, $f = g \upharpoonright \mu$. \square

p. 55, 3 times, 2nd line and 2nd and 3rd displayed equations:

Using Lemma 2.6.3, it is easily seen that f_0 is a function. Moreover, for each $\nu < \mu$, $f_0 \upharpoonright \nu = F(\nu)$, so for all $\nu < \mu$, we have

$$(\forall \alpha \in \nu)(f_0(\alpha) = h(\alpha, f_0 \upharpoonright \alpha)).$$

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha).$$

p. 56, 5 times, preceding Sect. 2.7:

$$(a) (\forall \alpha \in \text{On})(\exists y)(\forall z)[z = y \leftrightarrow \psi(\alpha, z)];$$

$$(b) (\forall \alpha)(\forall y)[\psi(\alpha, y) \leftrightarrow (\exists z)(z \text{ is a function} \wedge \text{dom}(z) = \alpha \wedge (\forall \xi \in \alpha)\phi(\xi, z \upharpoonright \xi, z(\xi)) \wedge \phi(\alpha, z, y))].$$

I shall not give the proof in detail. In fact, the idea is much as in Theorem 2.6.1, only now we cannot apply the replacement axiom to produce our function as we did then. Indeed, we cannot produce a *function* at all (working in ZF), since what we eventually get is a proper class. The only way to prove this is to start with the *formula* ϕ and explicitly produce an appropriate *formula* ψ as above.

We take for our ψ precisely the LAST formula that appears on the right of the double arrow in (b) above, namely,

$$(\exists z)(z \text{ is a function} \wedge \text{dom}(z) = \alpha \wedge (\forall \xi \in \alpha)\phi(\xi, z \upharpoonright \xi, z(\xi)) \wedge \phi(\alpha, z, y)).$$

This makes condition (b) trivially true and leaves us only to prove (a). (Actually, we should also check uniqueness, but this is really implicit in (b).) I sketch the proof, using classes instead of formulas.

Let $h : \text{On} \times V \rightarrow V$. Define a class f by

$$f = \{(\alpha, x) \mid (\alpha \in \text{On}) \wedge (\exists z)[(z \text{ is a function}) \wedge \text{dom}(z) = \alpha \wedge (\forall \xi \in \alpha)(z(\xi) = h(\xi, z \upharpoonright \xi)) \wedge x = h(\alpha, z)]\}.$$

It is easily seen that if $(\alpha, x), (\alpha, x') \in f$, then $x = x'$. And if there were an α such that no x existed with $(\alpha, x) \in f$, then consideration of the least such α would lead speedily to a contradiction. Hence $f : \text{On} \rightarrow V$. And clearly,

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha)$$

for all α . Finally, if $g : \text{On} \rightarrow V$ is such that $g(\alpha) = h(\alpha, g \upharpoonright \alpha)$, then by induction on α we get $f(\alpha) = g(\alpha)$ for all α , so $f = g$.

Exercise 2.6.2. Fill in the details in the above sketch. Then give the proof without any use of classes.

p. 62, 1 time, 3rd displayed equation:

$$K = \{f \mid (\exists g \in G)(f \subseteq g \upharpoonright C(g))\}.$$

p. 63, 1 time, 2nd displayed equation:

$$f(x) = h(x, f \upharpoonright x).$$

p. 94, 2 times, middle of page:

Then

$$f_\alpha \upharpoonright (\kappa_\alpha \times \{\alpha\}) : \kappa_\alpha \times \{\alpha\} \rightarrow \lambda_\alpha.$$

Since $\kappa_\alpha < \lambda_\alpha$ and $|\kappa_\alpha \times \{\alpha\}| = \kappa_\alpha$, the function $f_\alpha \upharpoonright (\kappa_\alpha \times \{\alpha\})$ cannot be surjective. Hence we can pick $\delta_\alpha \in \lambda_\alpha - f_\alpha[\kappa_\alpha \times \{\alpha\}]$. Let

$$\sigma = \langle \delta_\alpha \mid \alpha < \beta \rangle.$$

p. 110, 2 times, last displayed equation:

$$T_n = \{\epsilon \in {}^n 2 \mid (\forall m < n)[\epsilon \upharpoonright (m+1) \neq \epsilon_m \upharpoonright (m+1)]\}$$

p. 111, 3 times, following 3rd displayed equation:

Clearly, $f \in {}^\omega 2$. But for all n , $f \upharpoonright (n+1) \in T$, so $f \upharpoonright (n+1) \neq \epsilon_n \upharpoonright (n+1)$. Thus $f \notin \{\epsilon_n \mid n < \omega\}$, a contradiction. \square

p. 112, 10 times, bottom half of page:

Since the ordering is inclusion, we are only concerned with which sequences each T_α will contain. The definition is by recursion on the levels. That is, we define T_α from $\bigcup_{\beta < \alpha} T_\beta$. We use $T \upharpoonright \alpha$ to denote both the set $\bigcup_{\beta < \alpha} T_\beta$ and the tree on this set determined by the inclusion order. The recursion is carried out to preserve the following condition:

(*) If $s \in T_\alpha$ and $\alpha < \beta < \omega_1$, then, for each rational number $q > \sup(s)$, there is a $t \in T_\beta$ such that $s \subset t$ and $\sup(t) < q$.

To commence, we set $T_0 = \{\emptyset\}$. If $T \upharpoonright (\alpha+1)$ is defined, we define $T_{\alpha+1}$ as

$$T_{\alpha+1} = \{s \in {}^{\alpha+1}\mathbb{Q} \mid s \upharpoonright \alpha \in T_\alpha\}$$

where \mathbb{Q} is the set of rationals. If $|T_\alpha| \leq \aleph_0$, then since $|\mathbb{Q}| = \aleph_0$, we have $|T_{\alpha+1}| = \aleph_0$. Moreover, if (*) is valid for $T \upharpoonright (\alpha+1)$, it will clearly be valid for $T \upharpoonright (\alpha+2)$.

There remains the case where $T \upharpoonright \alpha$ is defined for α a limit ordinal. Let us call a branch b of $T \upharpoonright \alpha$ *cofinal* if it intersects each level of $T \upharpoonright \alpha$ (i.e. if its order-type under the tree-ordering is α). In order to define T_α , we must extend some cofinal branches of $T \upharpoonright \alpha$. Indeed, any element of T_α will necessarily be of the form $\bigcup b$, where b is a cofinal branch of $T \upharpoonright \alpha$.

p. 113, 8 times, preceding Sect. 4.5:

Now, if $\bigcup b \in T_\alpha$, $\bigcup b$ must be bounded above in \mathbb{Q} , so it must be the case that the set $\{\sup(s) \mid s \in b\}$ is bounded above in \mathbb{Q} . (As will become clear in a moment, it was in order to ensure that such branches b can always be found that we introduced the requirement (*).) Now, we cannot simply extend all such branches, since there are uncountably many of them, which would make T_α uncountable. On the other hand, we must ensure that (*) holds for $T \upharpoonright (\alpha+1)$. So we proceed as follows.

Notice first that (*) will hold for $T \upharpoonright \alpha$ providing it holds for each $T \upharpoonright \beta$ for $\beta < \alpha$.

Let $\langle \alpha_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal in α . For each $s \in T \upharpoonright \alpha$, and each rational number $q > \sup(s)$, we define an element $b(s, q)$ of ${}^\alpha \mathbb{Q}$ as follows.

Let $n(s)$ be least such that $s \in T \upharpoonright \alpha_{n(s)}$. By (*), pick $s_{\alpha(n)} \in T_{\alpha_{n(s)}}$ so that $s \subset s_{n(s)}$ and $\sup(s_{n(s)}) < q$.

We define $s(n)$ for $n(s) < n < \omega$ now by recursion. Let $s_{n+1} \in T_{\alpha_{n+1}}$ be such that $s_n \subset s_{n+1}$ and $\sup(s_{n+1}) < q$. If $\sup(s_n) < q$, then by (*), such an s_{n+1} can always be found.

Now set

$$b(s, q) = \bigcup_{n(s) < n < \omega} s_n.$$

Clearly, $b(s, q) \in {}^\alpha \mathbb{Q}$ and $s \subset b(s, q)$. Moreover, $\sup(b(s, q)) \leq q$. We define

$$t_\alpha = \{b(s, q) \mid s \in T \upharpoonright \alpha \wedge q \in \mathbb{Q} \wedge q > \sup(s)\}.$$

If $|T \upharpoonright \alpha| \leq \aleph_0$, then $|t_\alpha| \leq \aleph_0$. Moreover, $t \upharpoonright (\alpha+1)$ satisfies (*) by virtue of the construction.

That completes the definition of T . An easy induction on the levels shows that condition (ii) holds. (The induction steps have already been noted.) And (iii) follows directly from (*). The proof is complete. \square

p. 119, 4 times, 4th and 5th displayed equations:

$$g_\alpha \circ f_{A_1} \upharpoonright T = g_\alpha \circ f_{A_2} \upharpoonright T.$$

Since g_α is one-one, this implies that

$$f_{A_1} \upharpoonright T = f_{A_2} \upharpoonright T.$$

p. 179, 2 times, last line:

(For small systems M_0 .) It suffices to show that (ii) for small systems implies the unrestricted form of (ii). Let M_0 be a system, and let $\pi_1, \pi_2 : M_0 \rightarrow M$ be system maps. Let $a \in M_0$. Then $(M_0)_a$ is a small system, and $\pi_1 \upharpoonright (M_0)_a = \pi_2 \upharpoonright (M_0)_a$. In particular, $\pi_1(a) = \pi_2(a)$. But $a \in M_0$

p. 181, 1 time, proof of Lemma 7.8.13:

Since M is complete, G has a unique M -decoration, d . Let $d_0 = d \upharpoonright G_0$.

p. 186, $f \upharpoonright u$:

$f \upharpoonright u$, 14

p. 187, $T \upharpoonright \alpha$:

$T \upharpoonright \alpha$, 112