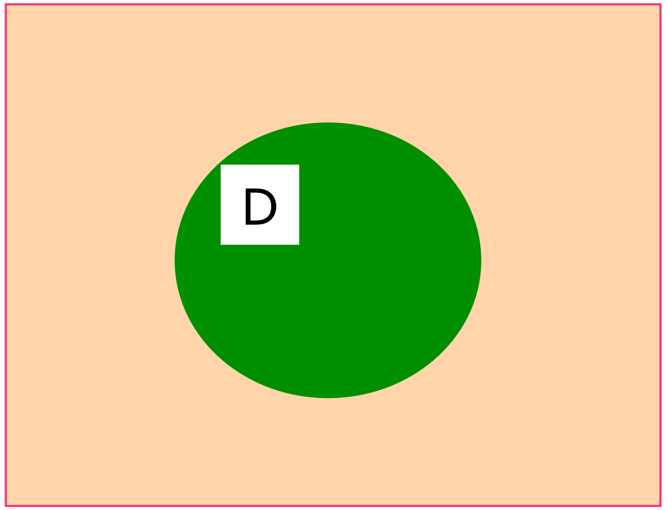
Axiomatic set theory

Jouko Väänänen

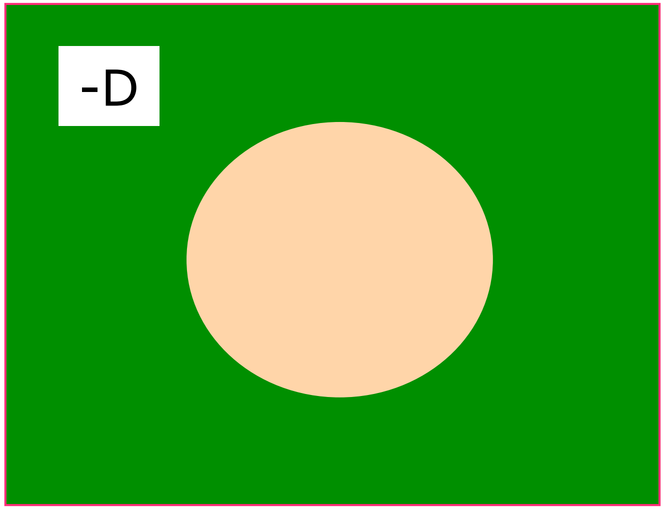
Set theoretical operations

- Union A∪B=elements that are in A or in B or in both
- Intersection A∩B=elements that are in A and in B
- Complement –A=elements that are not in A.
- x∈A means ”x is an element of A”
- A⊆B, A is a subset of B, means every element of A is an element of B.
- A=B, A and B are identical, means A and B have the same elements.

A set D

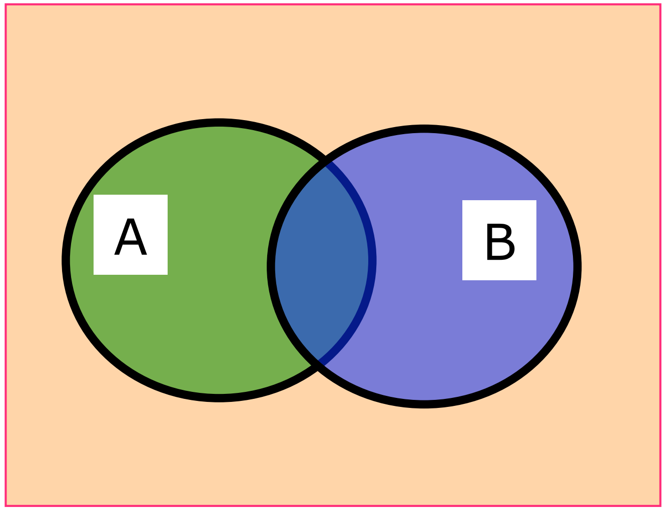


Its complement -D

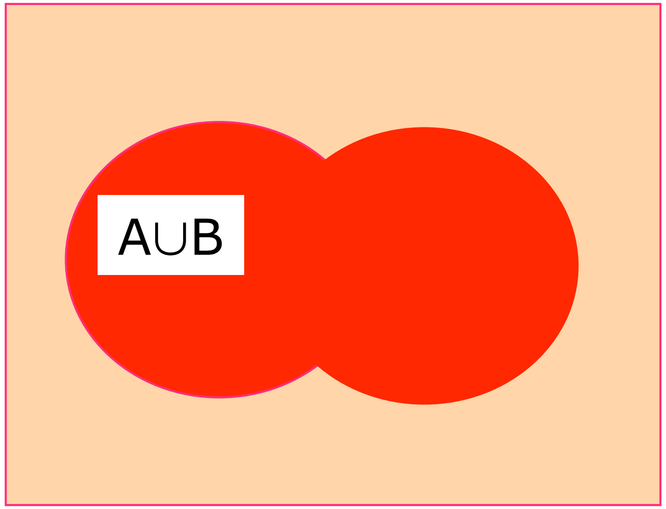


Last viewed

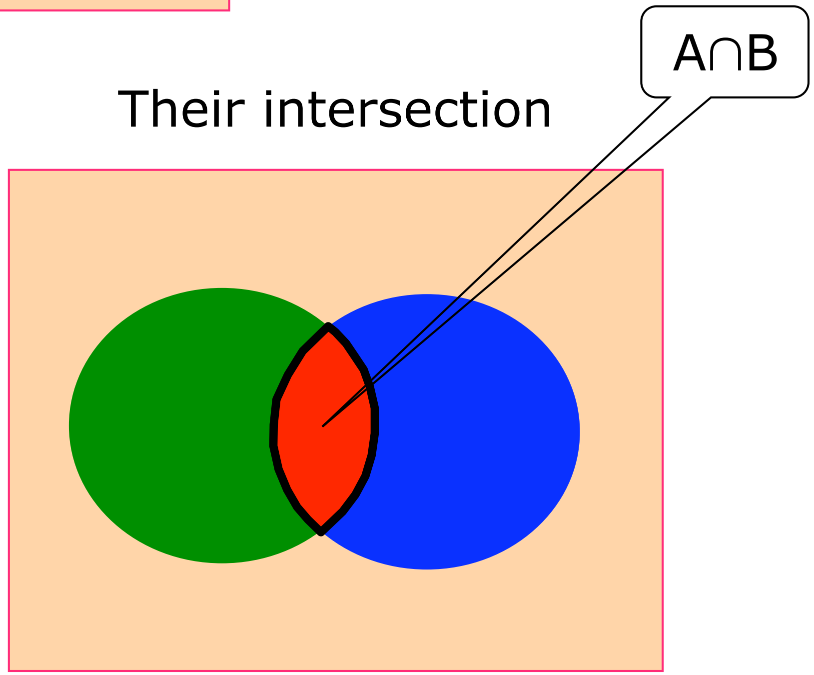
Two sets A and B



Their union



Their intersection



Examples

- To be proved:

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$$

$$z - (x \cup y) = (z - x) \cap (z - y)$$

Listing the elements of a set

- A set can be given by simply listing its elements in any order:
  - {0,1,2,3}
  - {0,2,4,6,...}
- Often a set is given by selecting elements with a certain property:
  - $\{x \in A : x < 10\}$

Special sets

- Empty set $\emptyset$, the set with no elements.
- Singleton set $\{a\}$ with a as its only element. E.g. $\{1\}$, $\{5\}$, $\{\emptyset\}$
- $\mathbb{N}=\{0,1,2,3,\ldots\}$
- Unordered pair $\{a,b\}$ with a and b as its only elements. E.g. $\{1,5\}$, $\{1,\{5\}\}$
- Note: $\{a,b\}=\{b,a\}$.

Union

- The union $\bigcup A$ of A is the set of all elements of elements of A
- $\bigcup\{a\}=a$
- $\bigcup\{a,b\}=a \cup b$
- $\bigcup\{a,b,c\}=a \cup b \cup c$
- $\bigcup\{a_i : i \in I\} = \bigcup_{i \in I} a_i$

Intersection

- The intersection $\cap A$ of A is the set of sets that are in all elements of A
- $\cap\{a\}=a$
- $\cap\{a,b\}=a \cap b$
- $\cap\{a,b,c\}=a \cap b \cap c$
- $\cap\{a_i : i\in I\}= \cap_{i\in I} a_i$

Examples

- To be proved

$$\bigcup_{i\in I}(a_i \cup b_i)=(\bigcup_{i\in I}a_i) \cup (\bigcup_{i\in I}b_i)$$

$$\bigcup_{i\in I}(a_i \cap b)=(\bigcup_{i\in I}a_i) \cap b$$

Power set

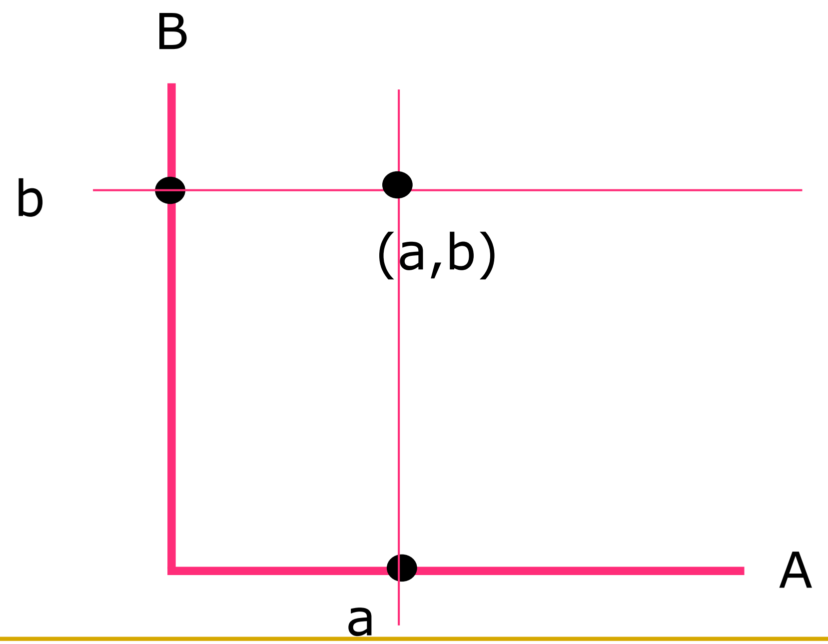
- The power set P(A) of A is the set of all subsets of A
- $P(\{a\})=\{\emptyset,\{a\}\}$
- $P(\{a,b\})=\{\emptyset,\{a\}, \{b\}, \{a,b\}\}$

Ordered pair

- The ordered pair of a and b is denoted (a,b)={a,{a,b}}.
- Exercise: (a,b)=(c,d) implies a=c and b=d.
- Ordered triple (a,b,c)=((a,b),c)
- Ordered n-tuple $(a_1,\ldots,a_n)$

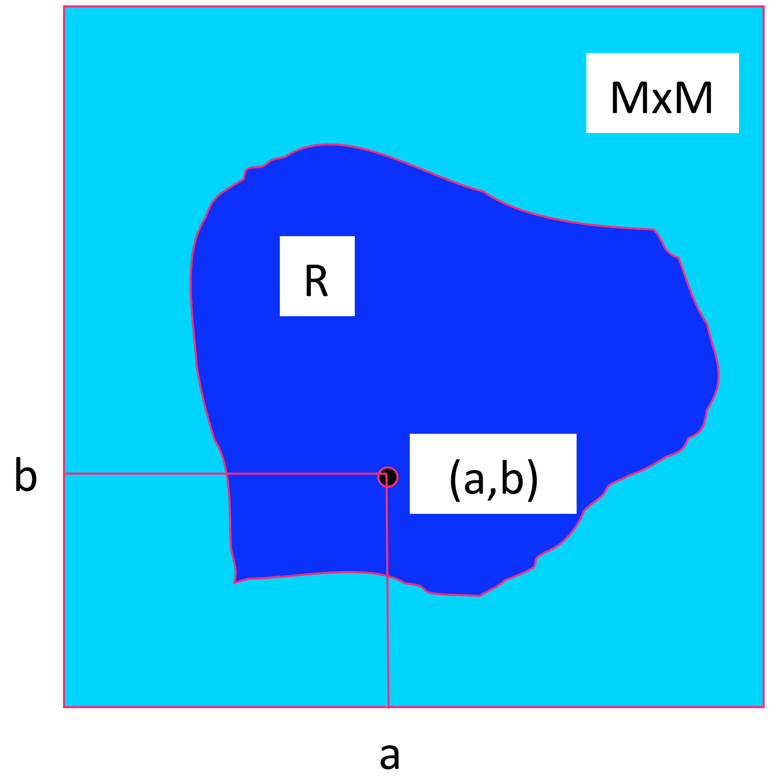
Cartesian product

- The Cartesian product A×B of A and B is the set of all ordered pairs (a,b) where a is in A and b is in B.



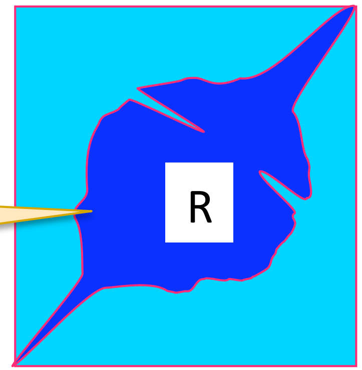
Relation

- A binary relation on a set M is any subsety R of the Cartesian product MxM.
- If (a,b) is in R, we write aRb, for simplicity

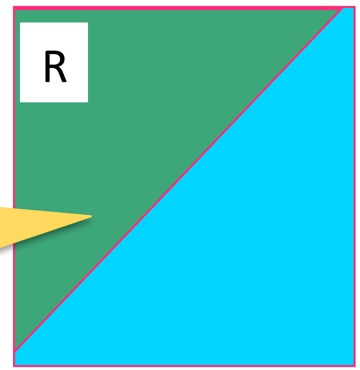


Properties of relations

- A binary relation is
  - Symmetric if aRb implies bRa
  - Reflexive if always aRa

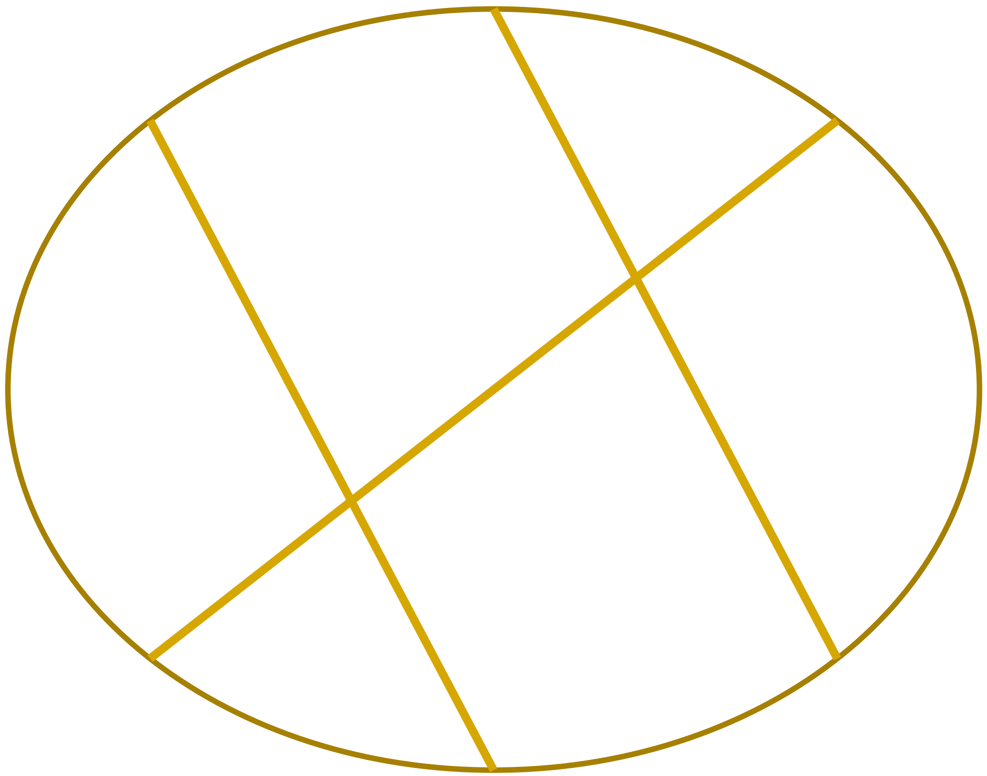


  - Transitive if aRb and bRc imply aRc
  - Antisymmetric if aRb & a≠b imply not bRa
  - Antrireflexive if never aRa



Last viewed

Equivalence relation



Function

- A set f of ordered pairs is a function, if for every $(x,b)\in f$ and $(x,c)\in f$ we have $b=c$. Then b is called the value of x under the function f, or the image of x under the unction f.
- The set of elements x with a value is called the domain of f, denoted dom(f).
- The set of values b is called the range of f, denoted ran(f).
- $f:x\rightarrow y$ means f is a function, $dom(f)=x$ and $ran(f)\subseteq y$.

Composition of functions

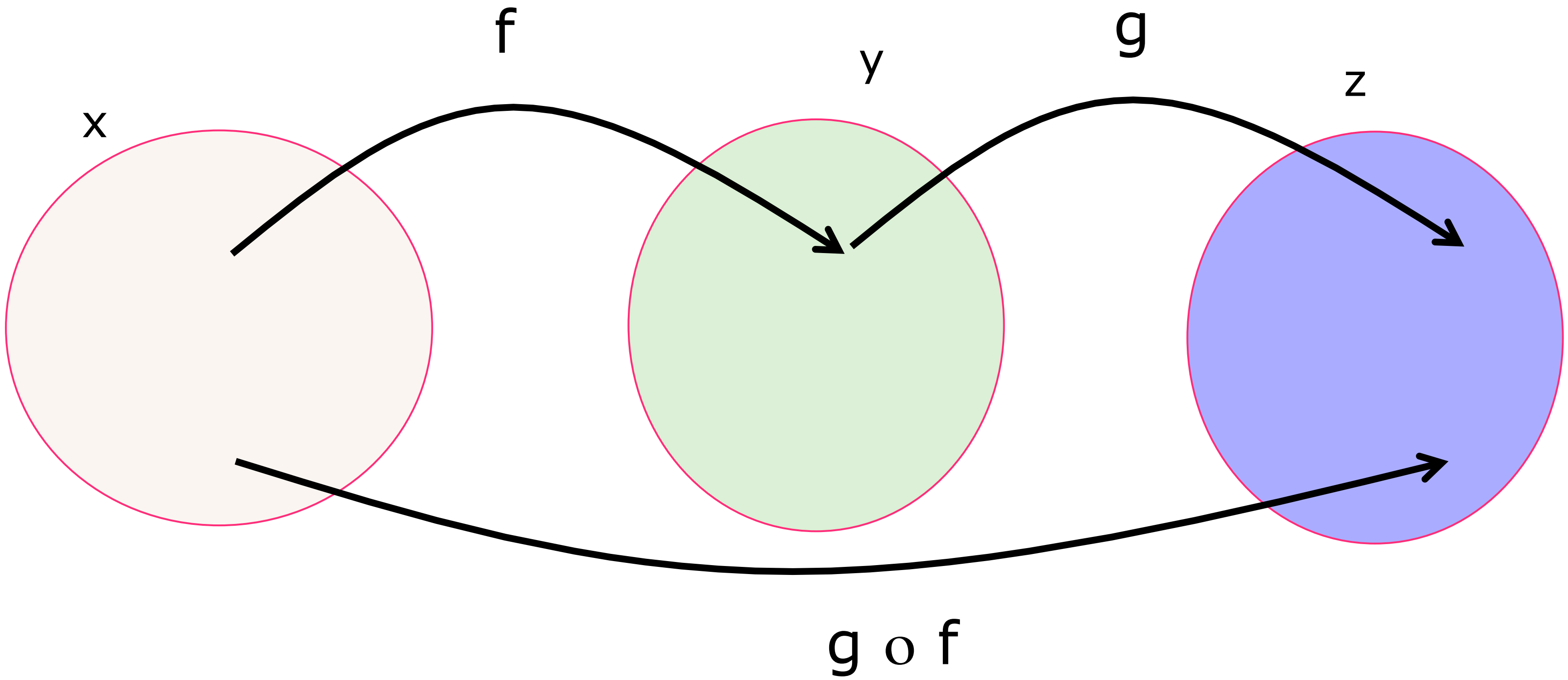
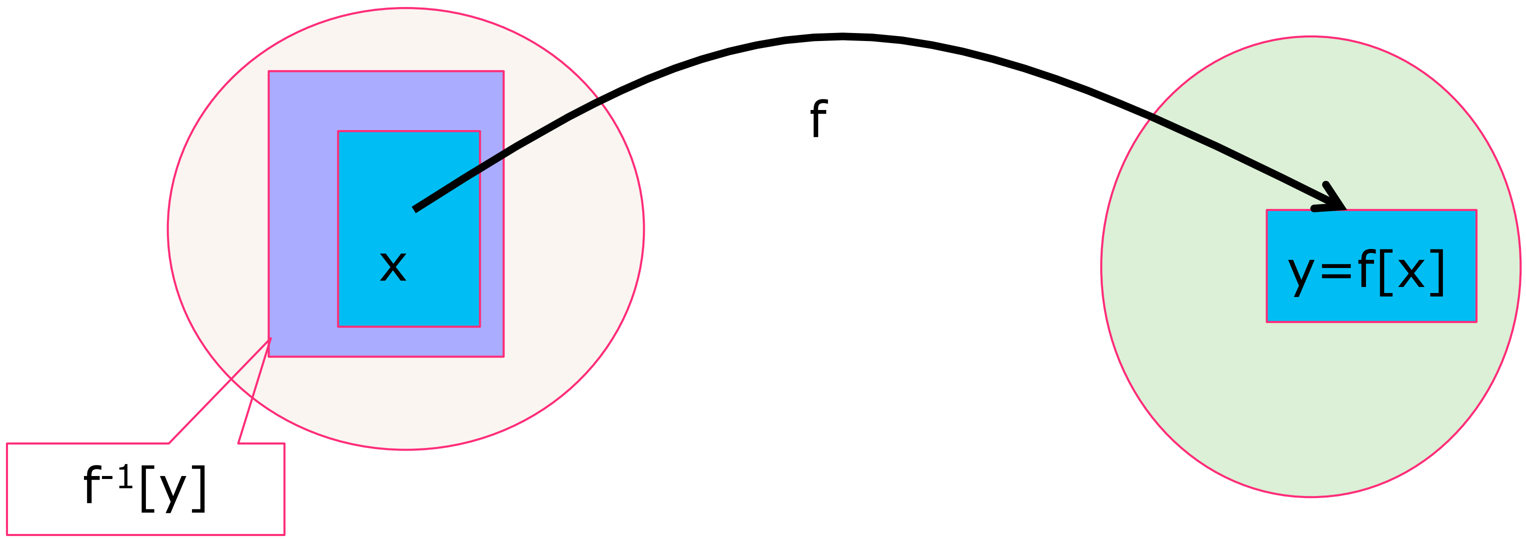


Image and preimage

- $f[x] = \{f(a): a \in x\}$
- $f^{-1}[x] = \{a : f(a) \in x\}$



Last viewed

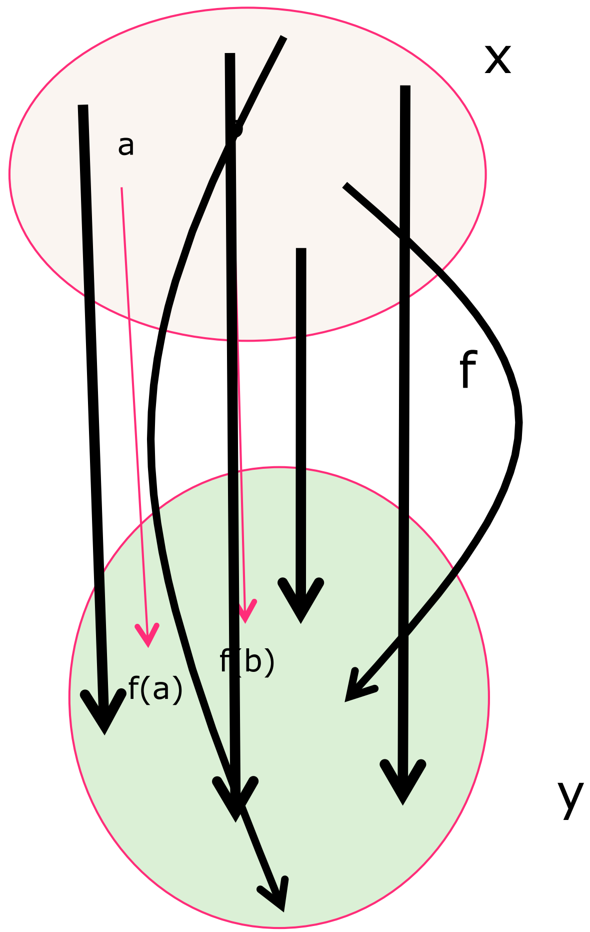
Example

- To be proved:

$$f^{-1}[\cup_{i\in I} a_i] = \cup_{i\in I} f^{-1}[a_i]$$

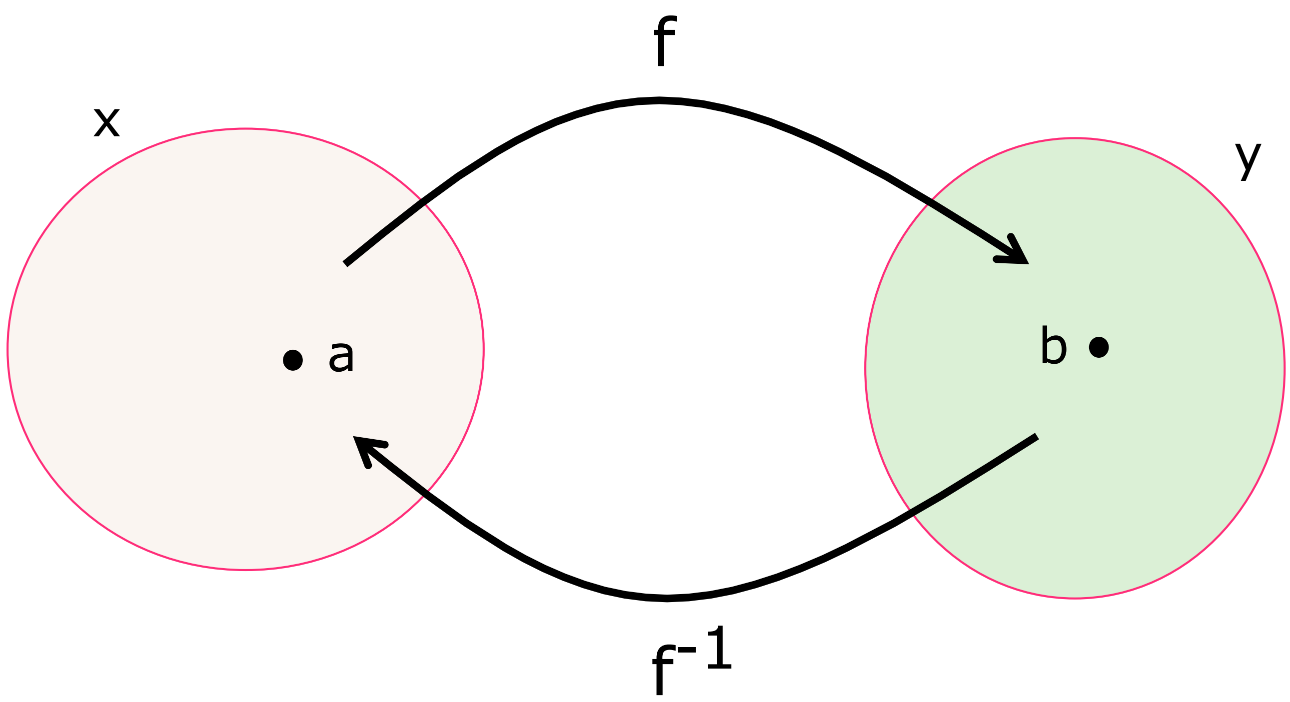
Properties of functions f:x→y

- f is injective (one to one): f(a)=f(b) → a=b
- f is surjective (onto): f[x]=y
- f is bijective: both injective and surjective



Inverse function

- A bijection has an inverse: $f^{-1}(a)=b$ iff $a=f(b)$



Last viewed

Generalized Cartesian product

- $\prod_{i\in I} x_i = \{f : (f:I\to\bigcup_{\in I} x_i)\ \&\ (\forall i\in I)(f(i)\in x_i)\}$
- $x^I = \prod_{i\in I} x$
- There is a canonical mapping between $\prod_{i\in\{1,2\}} x_i$ and $x_1 \times x_2$
- There is a canonical mapping between $\prod_{i\in\{1\}} x$ and x
- $x^\emptyset = \{\emptyset\}$