Axiomatic set theory

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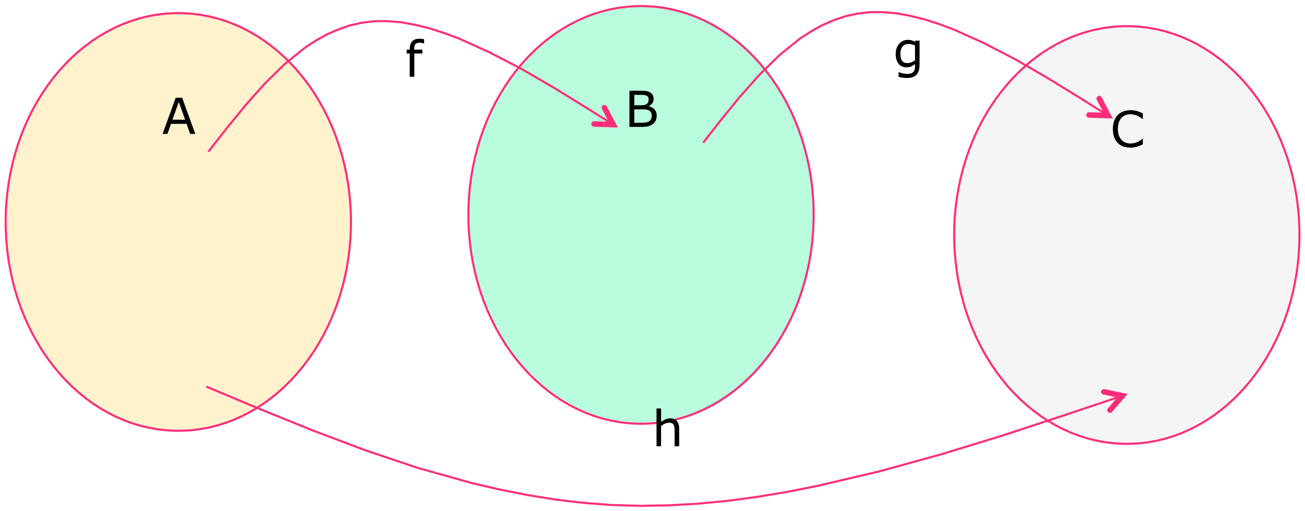
Equicardinality

A~B if there is a bijection f:A→B. Then A and B are equipollent.



1) A~A
2) A~B implies B~A
3) A~B and B~C imply A~C

Cantor's Theorem

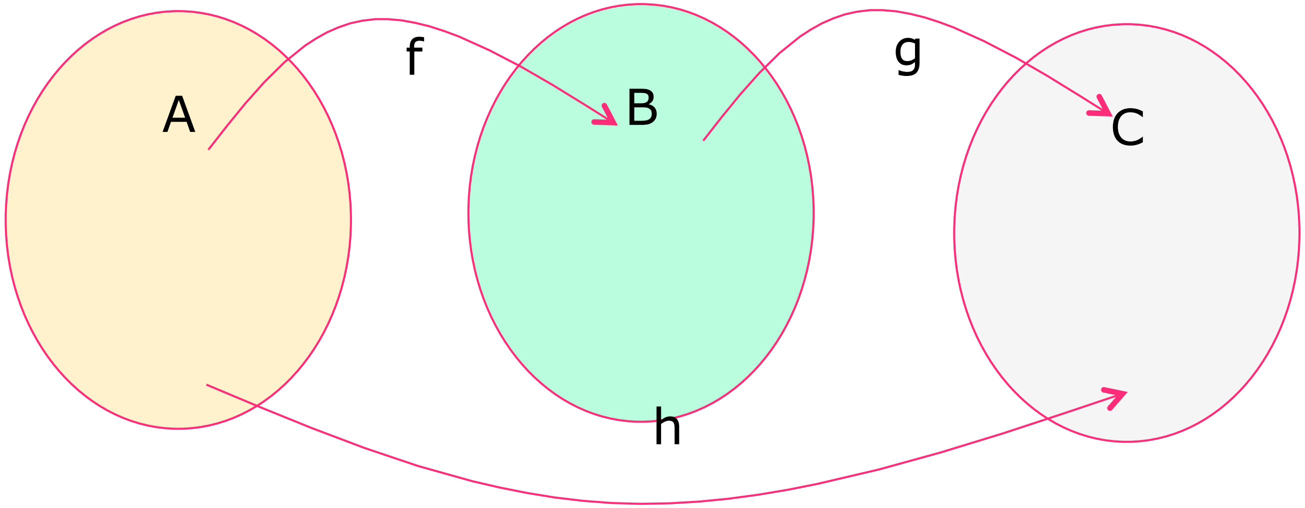
- There is no surjection $f:A \to P(A)$.
- Suppose there is.
- Let $X=\{a \in A : a \notin f(a)\}$. Suppose $X=f(a)$.
- If $a \in X$, then $a \notin f(a)=X$.
- So $a \notin X$. It follows that $a \in X$.
- A contradiction!

Another proof of Cantor's Theorem

- There is no injection $f:P(A)\to A$. (Note: then there is a surjection $A\to P(A)$, so we could use the previous proof, but we present a new proof anyway.)
- Suppose there is.
- $a_0=f(\emptyset)$.
- $a_1=f(\{a_0\})\neq a_0$
- etc
- $a_\omega=f(\{a_0,a_1,\ldots\})$.
- etc $a_\alpha=f(\{a_\beta : \beta<\alpha\})$
- No end! A contradiction!

Last viewed

Comparing cardinalities

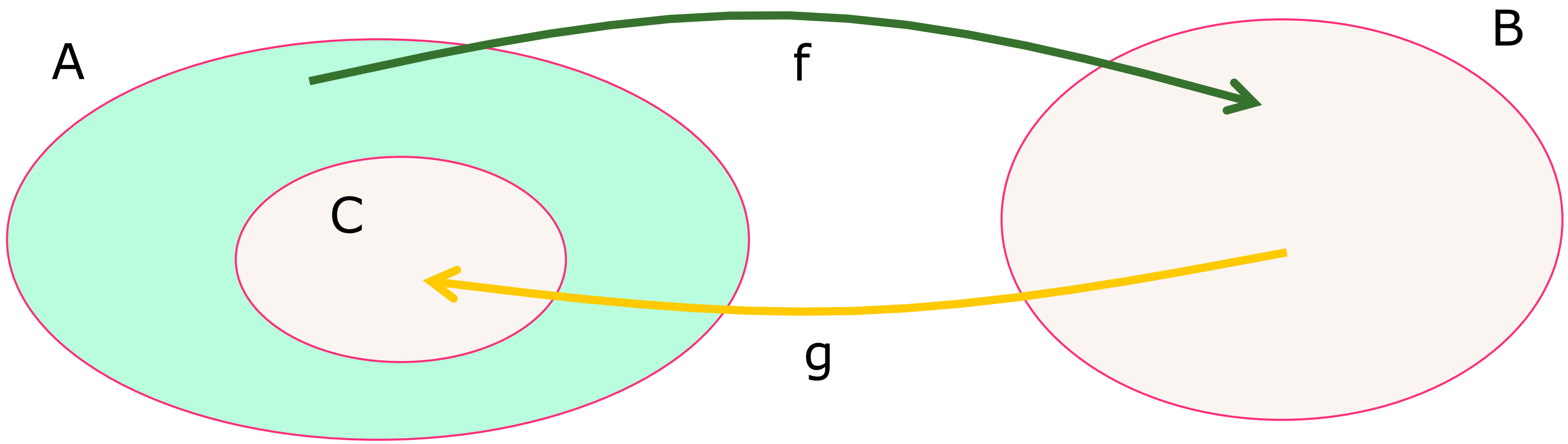


$|A| \leq |B|$ if there is a one to one mapping $f: A \to B$

1) $|A| \leq |A|$
2) $|A| \leq |B|$ and $|B| \leq |C|$ imply $|A| \leq |C|$

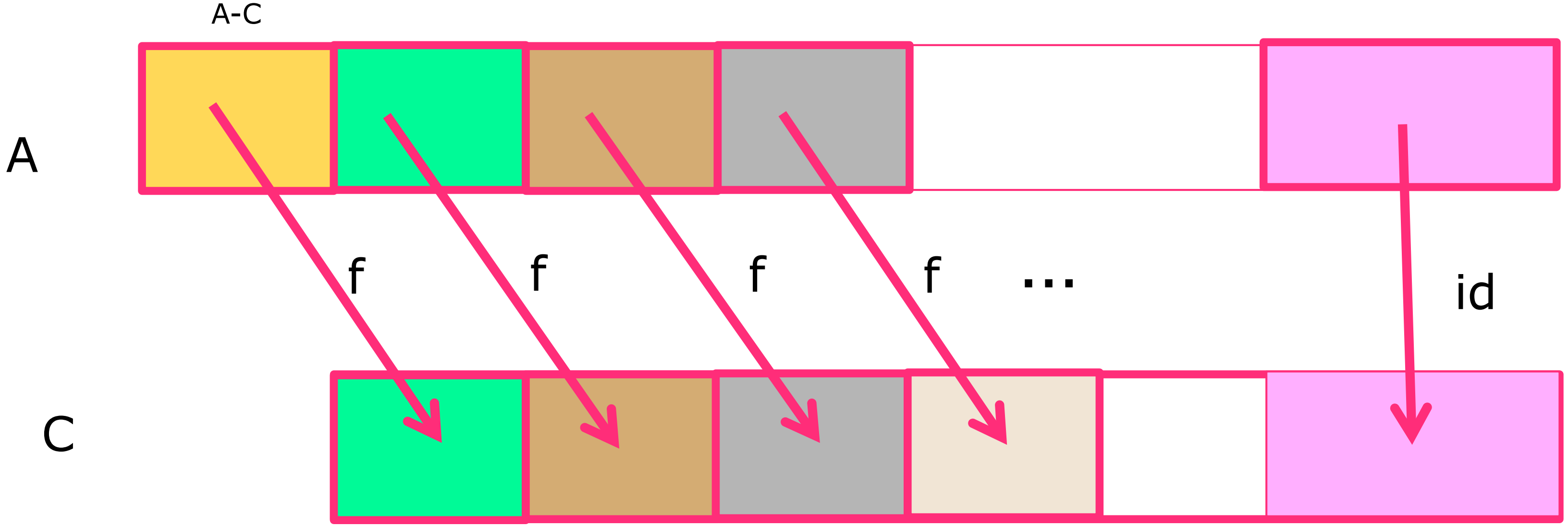
Cantor-Bendixson Theorem

- Suppose f:A→B and g:B→A are injections. Then there is a bijection h:A→B.
- Let C=g[B]. So g o f:A→C is an injection.
- If we find bijection h:A→C, then g^{-1} o h is a bijection A→B.



Construction of a bijection h:A→C

- Given injection f:A→C, C⊆A.



Cardinal numbers

- The cardinal number of a set A is the least ordinal α which is equipollent with A.
- An ordinal that is its own cardinal number is called a cardinal number or just a cardinal.
- Finite ordinals and ω are cardinals.

Successor and limit cardinals

- For every κ there is a bigger one (Cantors's Theorem!), denoted κ^+.
- These are the successor cardinals.
- Others are limit cardinals.
- κ is a limit cardinal iff $\forall\lambda<\kappa(\lambda^+<\kappa)$.

The aleph hierarch

- Aleph ℵ is the first in the Hebrew alphabet.
- Notation:
- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = (\aleph_\alpha)^+$
- $\aleph_\upsilon = \lim_{\alpha<\upsilon} \aleph_\alpha$
- $\aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_\omega < \aleph_{\omega+1} < \ldots$

Alternative notation

- $\aleph_\alpha$ is also denoted ω_α
- Then usually one emphasizes ω_α as an ordinal.
- $\omega_\alpha + \omega$ is the ordinal sum of ω_α and ω.
- $\aleph_\alpha + \omega$ is the cardinal sum of ω_α and ω.
- So let us define cardinal sum!!!

Cardinal sum

- Suppose κ_α are cardinals for $\alpha<\beta$.
- The cardinal sum $\Sigma_{\alpha<\beta}\,\kappa_\alpha$ is the cardinality of any set $\bigcup_{\alpha<\beta} A_\alpha$, where the sets A_α are disjoint and A_α has cardinality κ_α.
- Note: this is independent of the choice of the sets A_α. Any choice gives the same cardinality to the union.

Sum of two cardinals

- $\Sigma_{\alpha<2}\,\kappa_\alpha$ is written $\kappa_0+\kappa_1$.
- $\kappa + \lambda = \lambda + \kappa$
- $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$
- For finite n and m the cardinal sum n+m is the usual one, and the same as ordinal sum.
- If κ is infinite, then $\kappa+1=\kappa$ (Hilbert's hotel!).
- Hence: $\kappa + \lambda$ is different in cardinal arithmetic and in ordinal arithmetic.

Last viewed

Cardinal product

- Suppose κ_α are cardinals for $\alpha<\beta$.
- The cardinal product $\Pi^{\#}_{\alpha<\beta}\,\kappa_\alpha$ is the cardinality of any cartesian product $\Pi_{\alpha<\beta} A_\alpha$, where the sets A_α are disjoint and A_α has cardinality κ_α.
- Recall: $\Pi_{\alpha<\beta} A_\alpha$ is the set of functions f such that dom(f)=β and f(α)$\in A_\alpha$.
- Note: this is independent of the choice of the sets A_α. Any choice gives the same cardinality to the union.

Product of two cardinals

- $\Pi_{\alpha<2}\kappa_\alpha$ is written $\kappa_0 \cdot \kappa_1$.
- $\kappa \cdot \lambda = \lambda \cdot \kappa$
- $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$
- $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$
- For finite n and m the cardinal product n · m is the usual one, and the same as ordinal product.
- $\aleph_0 \cdot 2 = \aleph_0$ (even-odd trick!).
- Hence: $\kappa \cdot \lambda$ is different in cardinal arithmetic and in ordinal arithmetic.