Axiomatic set theory

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The shocking equation $\kappa \cdot \lambda = \kappa + \lambda$

- We prove $\kappa \cdot \lambda = \kappa + \lambda$ for infinite κ, λ
- Claim: $\kappa \cdot \kappa = \kappa$.
- Suppose not. Let κ be the smallest κ such that $\kappa \cdot \kappa > \kappa$. So $\lambda \cdot \lambda = \lambda$ for $\lambda < \kappa$.
- Let $P_\xi = \{(\alpha,\beta) : \alpha+\beta = \xi\}$.
- These sets are disjoint and cover $\kappa \times \kappa$.
- Define on P_ξ: $(\alpha,\beta) <_\xi (\alpha',\beta')$ if $\alpha < \alpha'$. Well-order!
- Define on $\kappa \times \kappa$: $(\alpha,\beta) < (\alpha',\beta')$ if $[(\alpha,\beta),(\alpha',\beta') \in P_\xi$ and $(\alpha,\beta) <_\xi (\alpha',\beta')]$ or $[(\alpha,\beta) \in P_\xi$ and $(\alpha',\beta') \in P_{\xi'}$ and $\xi < \xi'$. Well-order!

- Ord(κ x κ,<) has cardinality > κ
- So Ord(κ x κ,<) is isomorphic to an ordinal > κ
- So κ x κ has an element (α,β) such that its predecessors in < is a well-order isomorphic to κ.
- Suppose $(\alpha,\beta)\in P_\xi$.
- It is easy to see that the cardinality of the set of predecessors of (α,β) in < is at most $|\xi+1| \cdot |\xi+1|$.
- As $\lambda \cdot \lambda = \lambda$ for $\lambda<\kappa$, a contradiction!

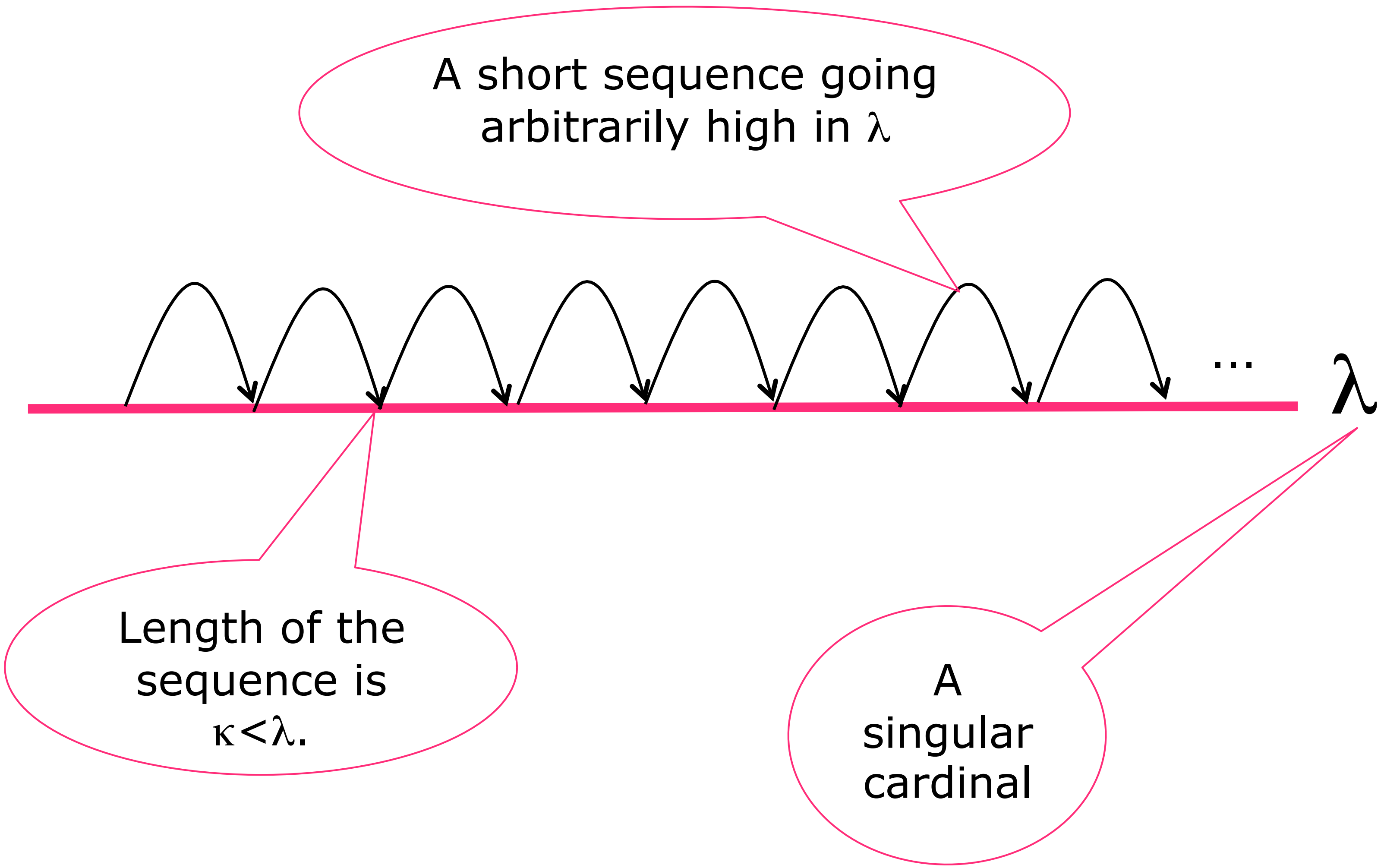
A proof of $\kappa \cdot \lambda = \kappa + \lambda$, for infinite κ, λ

- Claim: $\kappa \cdot \lambda = \kappa + \lambda = \max(\kappa,\lambda)$, for infinite κ, λ .
- Let $\mu=\max(\kappa,\lambda)\geq\omega$.
- $\kappa + \lambda \leq \kappa \cdot \lambda \leq \mu \cdot \mu = \mu \leq \kappa + \lambda$
- So the claim follows from the Schröder-Bernstein Theorem.
- The equation makes cardinal arithmetic easy. But exponentiation (later) is hard.

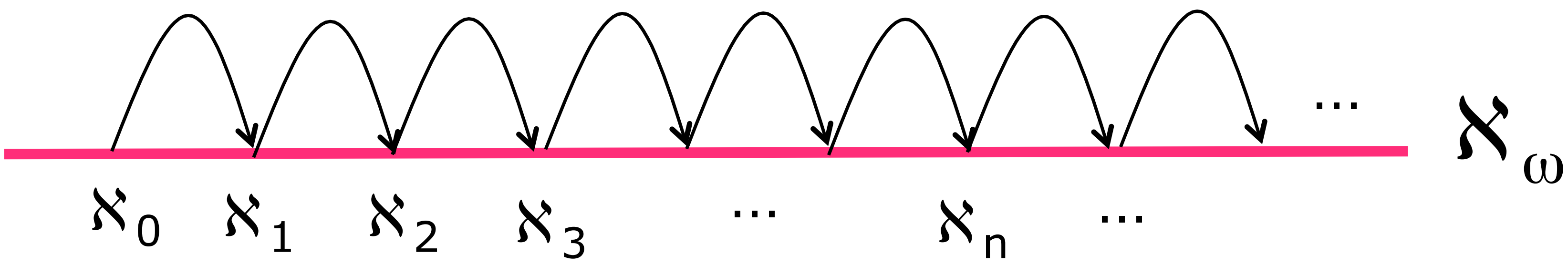
An important application

- Suppose κ is an infinite cardinal.
- The union of at most κ sets each of which has cardinality at most κ has itself cardinality at most κ.
- Proof: The union has clearly cardinality at most $\kappa \cdot \kappa$, i.e. κ.
- Corollary: The union of countably many countable sets is countable.

Regular and singular cardinals



An example: $\aleph_\omega$



Cofinality

- The cofinality $\mathrm{cf}(\delta)$ of a limit ordinal δ is the smallest ordinal α for which there is an increasing sequence $(\gamma_\beta)_{\beta<\alpha}$ of ordinals below δ such that the sequence is unbounded in δ.
- A cardinal κ is singular if $\mathrm{cf}(\kappa)<\kappa$, otherwise regular.

Examples

α	$cf(\alpha)$
ω	ω
$\omega+\omega$	
ω^2	
ω^ω	
ω_1	
$\omega_1+\omega_1$	
$\omega_1{}^2+\omega$	

Successor cardinals are regular!

- Claim: κ^+ is regular.
- Proof: Suppose the increasing sequence $(\gamma_\beta)_{\beta<\alpha}$ is unbounded in κ^+. We show that α cannot be $< \kappa^+$. Suppose it is. Then $|\kappa^+| = |\bigcup_{\beta<\alpha} \gamma_\beta| \leq |\alpha| \cdot \sup_{\beta<\alpha} |\gamma_\beta| \leq \kappa \cdot \kappa = \kappa$, a contradiction!

Examples

regular	singular
$\aleph_0$	$\aleph_\omega$
$\aleph_1$	$\aleph_{\omega+\omega}$
$\aleph_5$	$\aleph_{\omega_1}$
$\aleph_{\omega+1}$	$\aleph_{\omega_1+\omega}$
$\aleph_{\omega_1+\omega+3}$	
κ^+	$\kappa^+ + \kappa^{++} + \kappa^{+++} + \ldots$
κ^{++}	

Cardinal exponentiation

- Recall: $\lambda^\kappa = |\{f : f:\kappa \to \lambda\}|$
- Cliam: $2^\kappa = |P(\kappa)|$
- Proof: We define a bijection $g:\{f : f:\kappa \to 2\} \to P(\kappa)$.
- Let $g(f)=\{\alpha\in\kappa: f(\alpha)=0\}$
- Clearly, g is a bijection

  $\{f: f:\kappa \to 2 \} \to P(\kappa)$.
- Corollary: $2^\kappa > \kappa$ (by Cantor's Theorem)
- Corollary: $\lambda^\kappa > \kappa$, if $\lambda > 1$.

König's Theorem

- Suppose $\kappa_i<\lambda_i$ for all $i<\mu$.
- Then $\Sigma_{i<\mu}\ \kappa_i < \Pi_{i<\mu}\ \lambda_i$.
- Proof: Suppose F: $\cup_{i<\mu} A_i \rightarrow \Pi_{i<\mu}\ \lambda_i$, where the A_i are disjoint and $|A_i|=\kappa_i$. It suffices to show that F is **not** onto.
- Let $F_i(a)=F(a)(i)$. $F_i:A_i\rightarrow \lambda_i$ so F_i is not onto, as $\kappa_i<\lambda_i$, say it misses α_i.
- Let $h(i)=\alpha_i$. So h is in $\Pi_{i<\mu}\ \lambda_i$.

König's Theorem (Contd.)

- Claim: h is not in the range of F.
- Suppose h=F(a), say a is in A_i.
- Now α_i =h(i)=F(a)(i)=F_i(a), contrary to the choice of α_i.
- QED

Corollary to König's Theorem

- $\kappa^{cf(\kappa)} > \kappa$ (Note that we know already $\kappa^{\kappa} > \kappa$, as $2^{\kappa} > \kappa$, so now we improve this)
- Proof: Let $\alpha = cf(\kappa)$. So $\kappa = \bigcup_{\beta<\alpha} \gamma_\beta$, where $\gamma_\beta < \kappa$. Let $\gamma'_\beta = \gamma_\beta \setminus \bigcup_{\eta<\beta} \gamma'_\eta$, and $\kappa_\beta = |\gamma'_\beta| < \kappa$. So $\kappa = \Sigma_{i<\alpha} \kappa_i < \Pi_{i<\alpha} \kappa = \kappa^{cf(\kappa)}$.
- QED

How many real numbers?

- $2^{\aleph_0}$
- By Cantor's Theorem, at least $\aleph_1$
- Exactly $\aleph_1$? Continuum Hypothesis!
- By König's Theorem, $2^{\aleph_0}$ is not a countable sum of smaller cardinals.
- This is all that can be said on the basis of ZFC.

The picture

$\kappa=\kappa^2=\kappa^3=\ldots$

$\kappa^\omega>\kappa$, if $cf(\kappa)=\omega$, otherwise $\kappa^\omega=\kappa$ is possible

$\kappa^{\omega_1}>\kappa$, if $cf(\kappa)\leq\omega_1$, otherwise $\kappa^{\omega_1}=\kappa$ is possible

Etc, etc