Axiomatic set theory

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Borel sets

- We consider sets of real numbers, that is sets $A \subseteq \mathbb{R}$.
- A is open if for every a in A there is an n such that $(a-1/n, a+1/n) \subseteq A$.
- The meaning of openness is that to know whether a is in A or not we need to know only an approximation of a, for example 5 first decimals.
- Examples: $(0,1)$, $(0,1) \cup (5,9)$, $(-2, \infty)$, $\mathbb{R}$, $\emptyset$

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Closed sets

- A set is closed if its complement in $\mathbb{R}$ is open.
- The meaning of closedness is that to know that a is in A we need only know that a is the limit of elements in A.
- Examples: $[0,1]$, $[0,1]\cup[5,9]$, $[-2, \infty)$, $\{-1,4,70\}$, $\mathbb{R}$, $\varnothing$
- A closed set in the plane: $\{(x,y): x^2+y^2=1\}$.

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Borel sets

- Borel sets are sets which are built up from open intervals by means of simple operations, like $A-B$, $A\cup B$, $A\cap B$, etc
- Examples: $[0,1)$, $(0,1]\cup[5,9)$, $\{x:x\neq 0$ and $x\neq 1\}$.

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σ-algebras

- A σ-algebra is a non-empty set $\mathcal{B}$ of sets ($A \subseteq \mathbb{R}$) such that $\mathcal{B}$ is closed under complement $-B$ and countable union $\cup_n A_n$.
- Every σ-algebra is also closed under countable intersection $\cap_n A_n$.
- Examples: $\{\emptyset, \mathbb{R}\}$, $\{A : A \subseteq \mathbb{R}\}$

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Easy lemma

- The intersection of any family of σ-algebras is again a σ-algebra.
- Corollary: If $\mathcal{A}$ is a family of sets, there is the smallest σ-algebra $\mathcal{B}$ containing $\mathcal{A}$, namely the intersection of all σ-algebras containing $\mathcal{A}$.

Definition of Borel sets

- The family $\mathcal{B}$ of Borel sets is the smallest σ-algebra containing the family of all open intervals (a,b).
- Fact: Every open set is a Borel set.
- Proof: Suppose V is open. Then V is the union of the family of all open intervals (a,b) contained in V, where a and b is a rational number. To see why, suppose x is in V. Since V is open, there is a neighborhood (a,b) of x contained in V. Since the rationals are dense in the reals, there are rational numbers a′ and b′ such that $x\in(a',b')\subseteq(a,b)\subseteq V$. qed

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Construction of Borel sets

- We can reconstruct $\mathcal{B}$ by transfinite induction as follows:
- Let $\mathcal{B}_0$ be the set of open intervals (a,b)
- Suppose $\mathcal{B}_\alpha$ has been defined and α is a countable ordinal. Let $\mathcal{B}_{\alpha+1}$ consist of countable intersections of members of $\mathcal{B}_\alpha$, countable unions of members of $\mathcal{B}_\alpha$, and complements -B of members of $\mathcal{B}_\alpha$
- $\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$ if λ is a countable limit ordinal

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$\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$

- We first show that $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ is a σ-algebra.
- $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ is closed under set difference A-B: Suppose $A \in \mathcal{B}_\alpha$. Let $\gamma=\alpha+1$. Then

  $-A \in \mathcal{B}_\gamma$.
- $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ is closed under countable unions $\bigcup_n A_n$. Suppose $A_n \in \mathcal{B}_{\alpha_n}$. Let $\alpha=\sup_n \alpha_n$. Note that $\alpha<\omega_1$. Now $\bigcup_n A_n \in \mathcal{B}_{\alpha+1}$.

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Contd.

- Now $\mathcal{B} \subseteq \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ follows as $\mathcal{B}$ is the smallest σ-algebra containing all open intervals.
- On the other hand, we prove $\mathcal{B}_\alpha \subseteq \mathcal{B}$ for all α. But this follows by easy induction from the definition of σ-algebras.
- Conclusion: $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$

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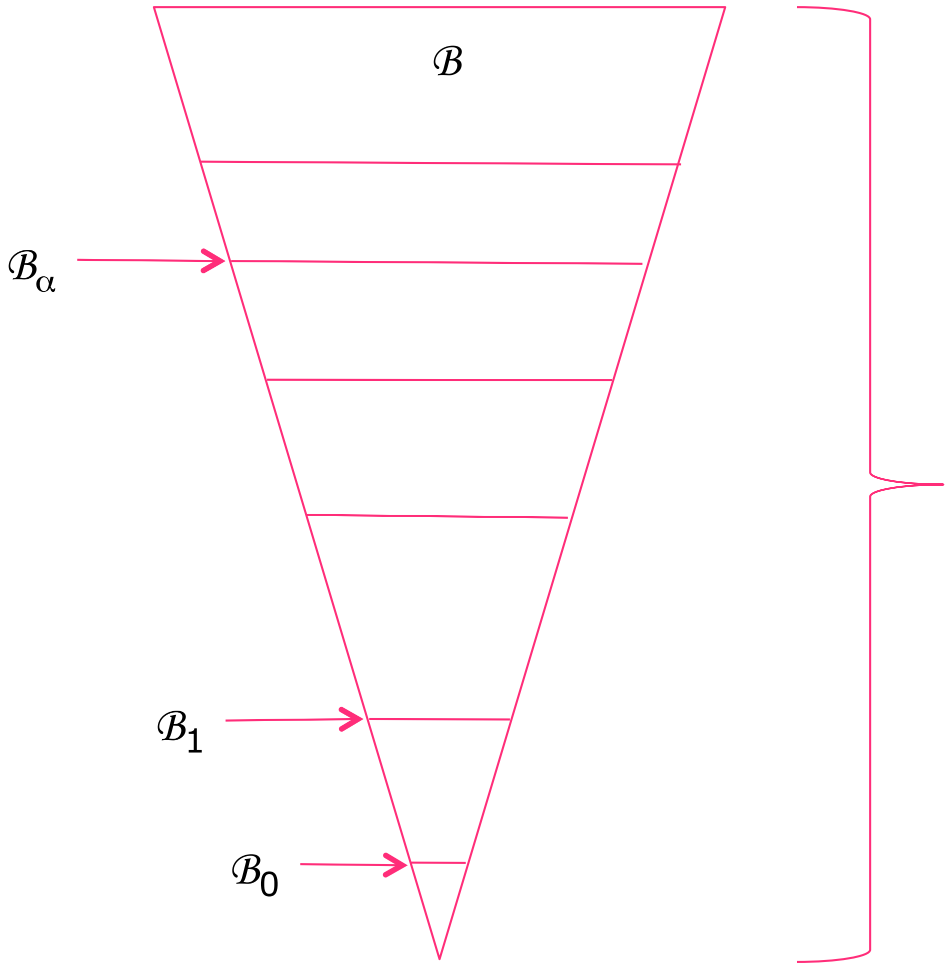
Punchline

- The family of Borel sets has two definitions which coincide:
  - The original definition as the smallest σ-algebra containing all open intervals.
  - The second definition as the union of the hierarchy of the sets $\mathcal{B}_\alpha$, $\alpha<\omega_1$.
- So we can prove things for all Borel sets by using transfinite induction on α.

Picture



The height is ω_1

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How many Borel sets are there?

- The nice thing about Borel sets is that there total number is just 2^ω, while the number of all sets (of reals) is 2^{2^ω}.
- Claim: $|\mathcal{B}| = 2^\omega$
- Proof: $|\mathcal{B}_\alpha| = 2^\omega$ is proved by an easy induction on α. From this $|\mathcal{B}| = 2^\omega$ follows, as $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$.

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