Axiomatic set theory

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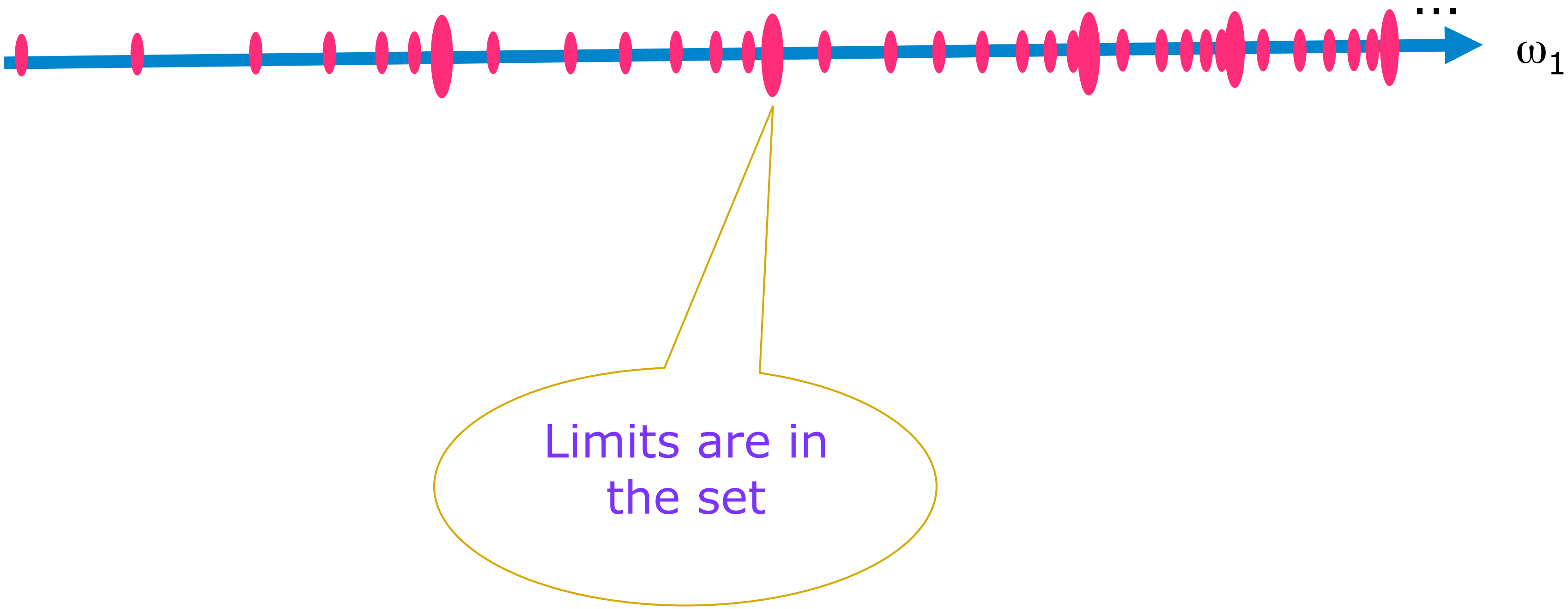
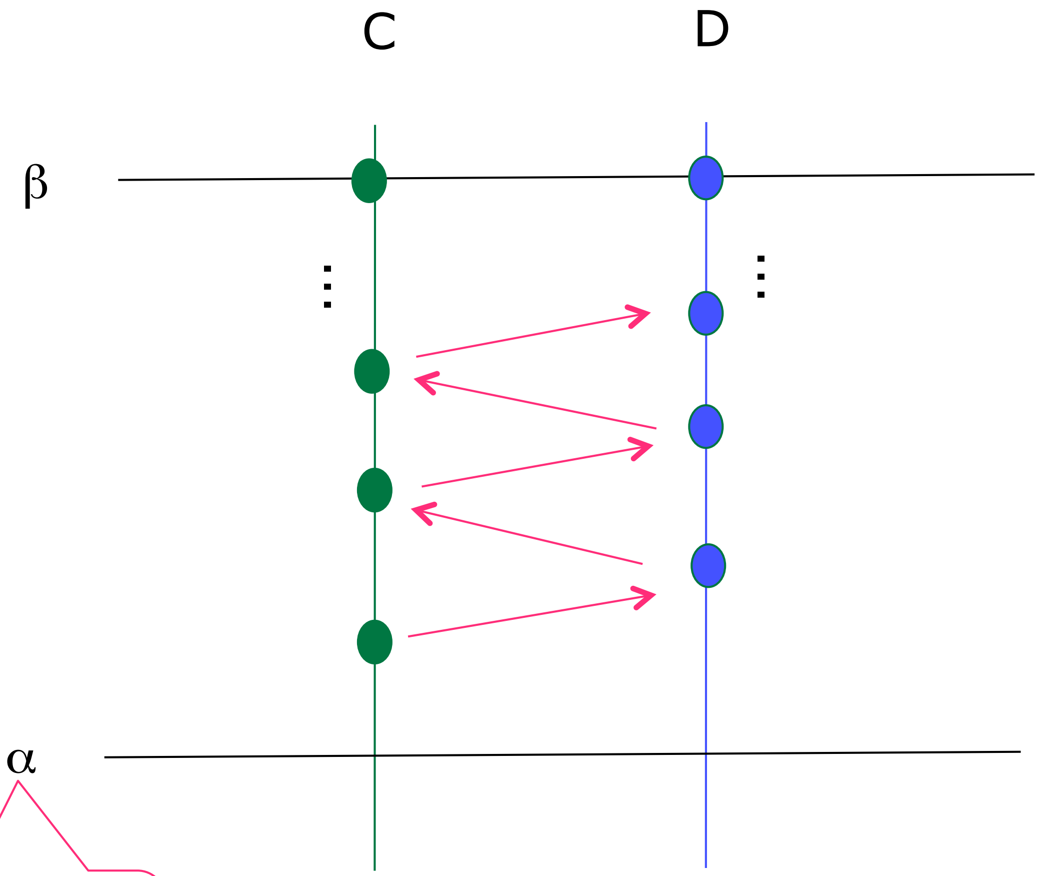
Closed unbounded sets

- We consider sets C of countable ordinals.
- C is closed if C contains every countable ordinal α for which $C \cap \alpha$ is unbounded in α.
- C is unbounded if it is uncountable.
- C is club (closed unbounded) if it is both closed and unbounded.
- Examples: The set of countable limit ordinals is club, the set of countable successor ordinals is not.

Picture of a club set (pretty hard to draw, though)



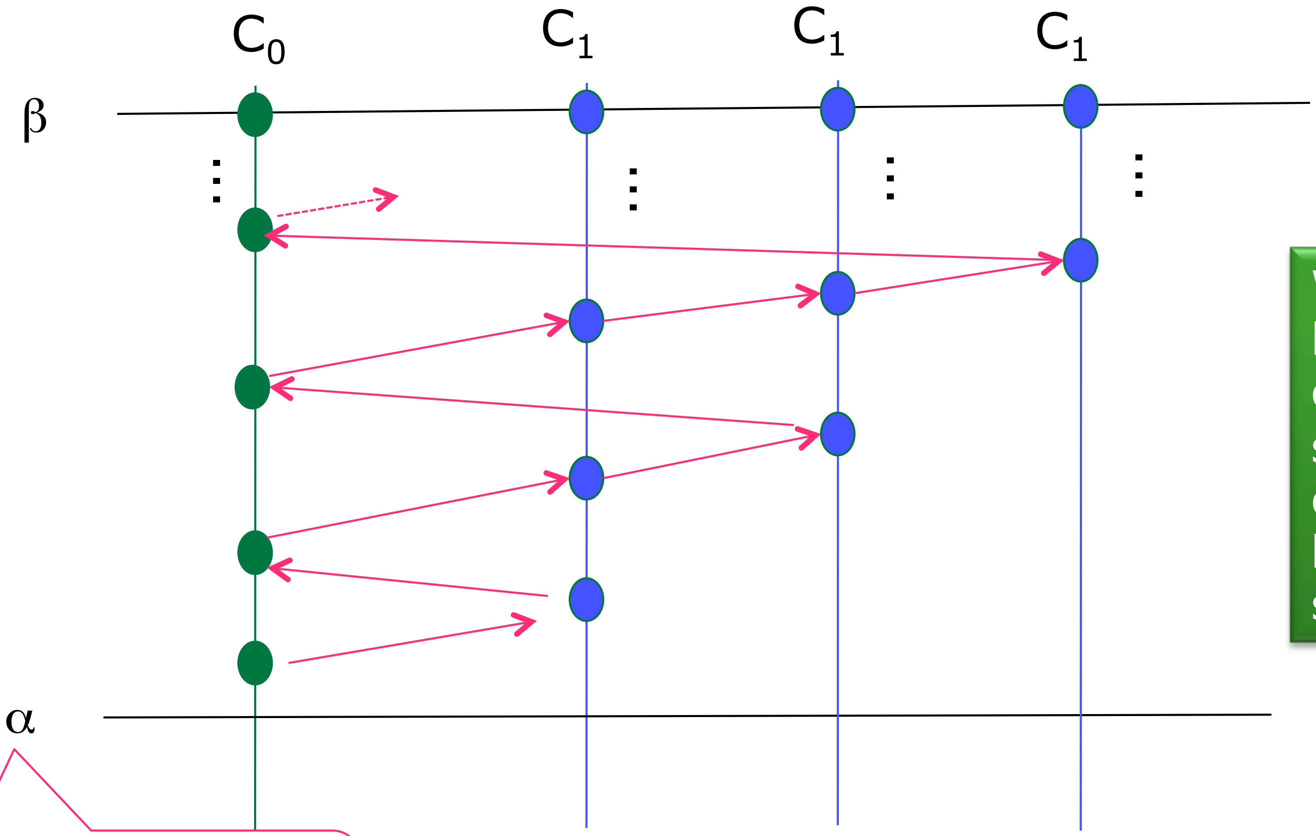
The intersection of two club sets is club



We use unboundedness of both C and D to go zick zack between C and D. By closedness, the limit is in both sets.

We want an element β greater than α in the intersection

The intersection of countably many club sets is club



We go zick zack between all the countably many sets By closedness, the limit is in each set.

We want an element β greater than α in the intersection

Last viewed

How long intersections of clubs are clubs?

- Suppose C_α is club for each $\alpha<\omega_1$.
- $\{\alpha : \alpha\in C_0\cap C_1\cap\ldots\cap C_n\}$ is club for all n
- $\{\alpha : \forall n\in\omega(\alpha\in C_n)\}$ is club
- $\{\alpha : \forall\beta\in\omega+\omega\ (\alpha\in C_\beta)\}$ is club
- ...
- what about the "diagonal intersection":
- $\{\alpha : \forall\beta\in\alpha\ (\alpha\in C_\beta)\}$ is club? Yes!

Diagonal intersection of clubs is a club.

- Suppose C_α is club for each $\alpha<\omega_1$.
- Let $C=\{\alpha : \forall\beta<\alpha\ (\alpha\in C_\beta)\}$.
- We show that C is club
- 1st: C is closed:
- Suppose α is a limit point of C, say α is the limit of the increasing sequence (α_n).
- We show that α is in C. Let $\beta<\alpha$.
- We have to show that $\alpha\in C_\beta$. Choose m such that $\beta<\alpha_m$. Since $\alpha_n\in C$, $\alpha_n\in C_\beta$ for all n>m. Since C_β is closed, $\alpha\in C_\beta$. QED

Contd.

- C is unbounded:
- Suppose δ is given. Let $\alpha_0 \in \bigcap_{\beta<\alpha} C_\beta$ such that $\alpha_0 > \delta$. Let $\alpha_1 \in \bigcap_{\beta<\alpha_0} C_\beta$, $\alpha_2 \in \bigcap_{\beta<\alpha_1} C_\beta$, etc. so that the sequence is strictly increasing.
- Finally, let α be the limit of (α_n).
- We show that β is in C, i.e. $\forall\beta<\alpha\ (\alpha \in C_\beta)$. Let $\beta<\alpha$. Choose m such that $\beta<\alpha_m$. So $\alpha_n \in C_\beta$ when $n>m$. Thus α is in C_β. QED

Stationary sets

- A set S of countable ordinals is stationary if it meets every club set.
- Examples: ω_1 is stationary,
- … as is the "tail set" $\omega_1 - \alpha$ for an α.
- The set of limit ordinals is stationary,
- … as is the set of limits of limit ordinals, because every club set contains a limit ordinal and also a limit of limit ordinals.
- The set of successor ordinals is not stationary, because it does not meet the club set of all limit ordinals.
- Every club sets is stationary.

A question arises...

- Are all stationary sets club? We shall see that they are not.
- So stationary sets are ”large” but not as large as club sets.

Pressing Down (Fodor) Lemma

- Suppose $f:S\rightarrow \omega_1$ is pressing down, i.s. $f(\alpha)<\alpha$ for all α in S. If S is stationary, then there is stationary $S^*\subseteq S$ such that f is constant on S^*.
- Proof: Suppose this is not the case.
- Then for all α the set $\{\beta:f(\beta)= \alpha\}$ is non-stationary.
- Let C_α be a club that does not meet $\{\beta:f(\beta)= \alpha\}$.
- Let $\delta\in S$ belong to the diagonal intersection of the C_α i.e. $\forall\beta\in\delta\ (\delta\in C_\beta)$.
- Let $\beta=f(\delta)$. So $\beta<\delta$, as f is pressing down.
- Thus $\delta\in C_\beta$ and because of the way C_β was chosen, $f(\delta)\neq \beta$, a contradiction.

Informal interpretation of pressing down lemma:

- Everybody knows one cannot maarch down in infinite descending chain of ordinals.
- Fodor lemma says more.
- You cannot even go systematically ”one step down”, even if you skip some ordinals, unless you make a constant jump!

There are stationary sets that are not club

- Let for each countable limit ordinal α an increasing sequence (α_n) be chosen so that (α_n) converges to α.
- Consider the regressive function which maps α to α_n. This is constant on a stationary set S_n.
- If each S_n were club, then their intersection would be club.
- But the intersection can contain only **one** ordinal!
- So at least one of the stationary sets S_n fails to be club. QED

Overwiew of advanced set theory

- The study of stationary sets is part of combinatorial set theory.
- Inner models are subclasses of V which still satisfy all the axioms of ZFC. By means of inner models one can show e.g. that the Continuum Hypothesis is consistent with ZFC.
- There is a method called forcing for contructing extensions of models of set theory. Using forcing one can show e.g. that Continuum Hypothesis is unprovable in ZFC