Axiomatic set theory

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Notation

- Sets: capital letters X,Y,...
- Elements: small letters x,y,...

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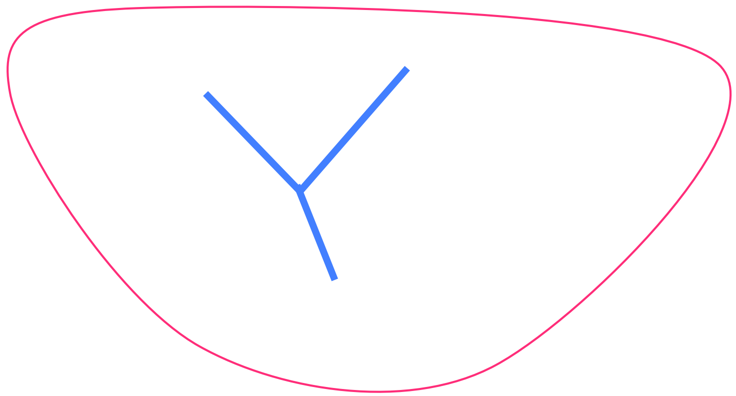
Partial order (X,≤), poset

- Reflexive, antisymmetric and transitive
- Example (P(X), ⊆)

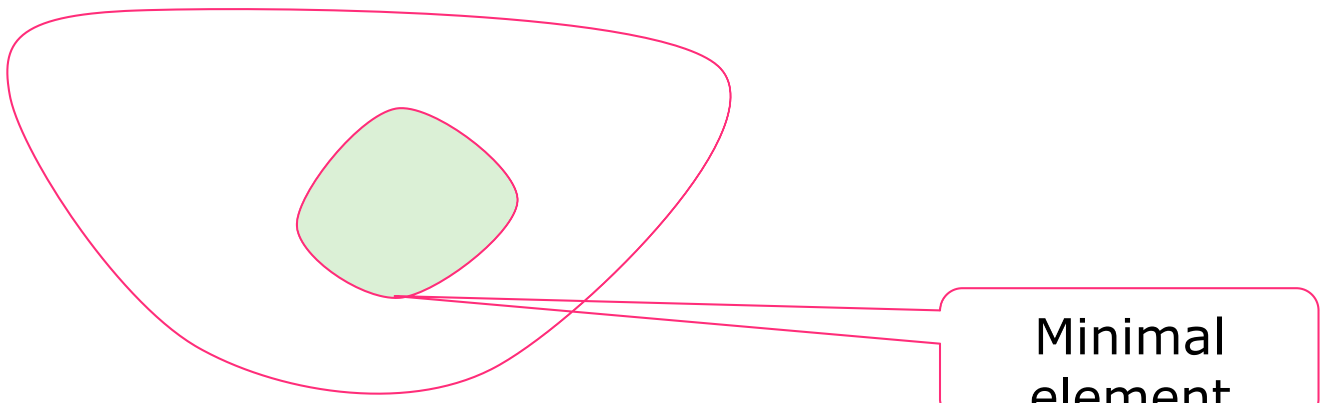


Lemma

- Suppose $(X, \leq)$ is a poset. Then there is a set Y such that $(X, \leq) \cong (Y, \subseteq)$.
- Proof: (Lecture)

Well-founded poset

- Minimal element: none smaller
- Partial order is well-founded if every non-empty subset has a minimal element.



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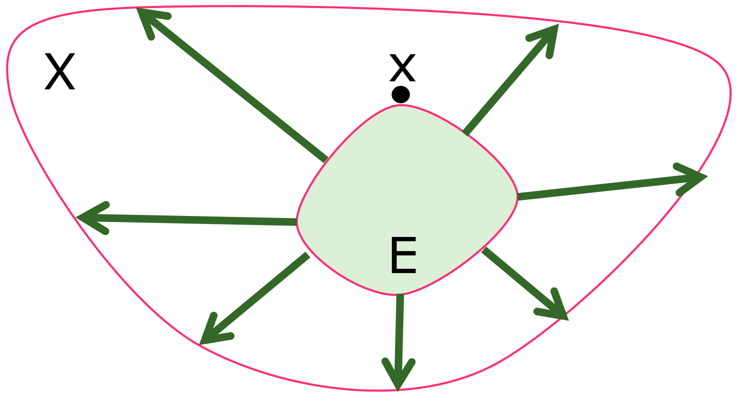
Lemma

- A poset is well-founded iff it has no infinite descending sequence.
- Proof: (lecture)

Induction on a well-founded set

- Suppose $(X,\leq)$ is a well-founded poset.
- Suppose $E\subseteq X$ is such that
  - If $x\in X$ and every $y\in X$ with $y<x$ is in E, then $x\in E$.
- Then $E=X$.



Proof

- Suppose X-E$\neq\emptyset$.
- Let x be <-minimal element in X-E.
- Then every y<x is in E.
- Hence by assumption x$\in$E, a contradiction.
- So E=X. QED

Total (linear) ordering

- Poset that is connected i.e. if $a \neq b$, then $a \leq b$ or $b \leq a$.

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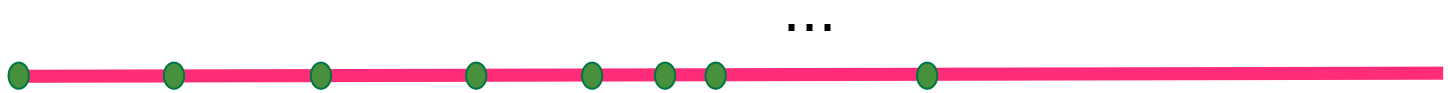
Wellordering, wellordered set, woset

- Linear ordering that is well-founded.



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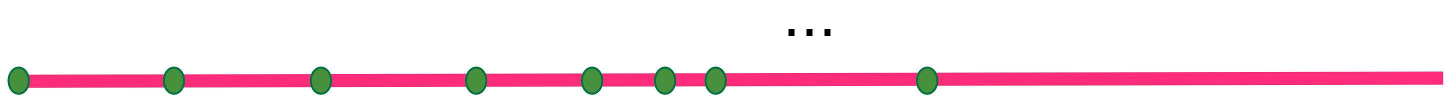
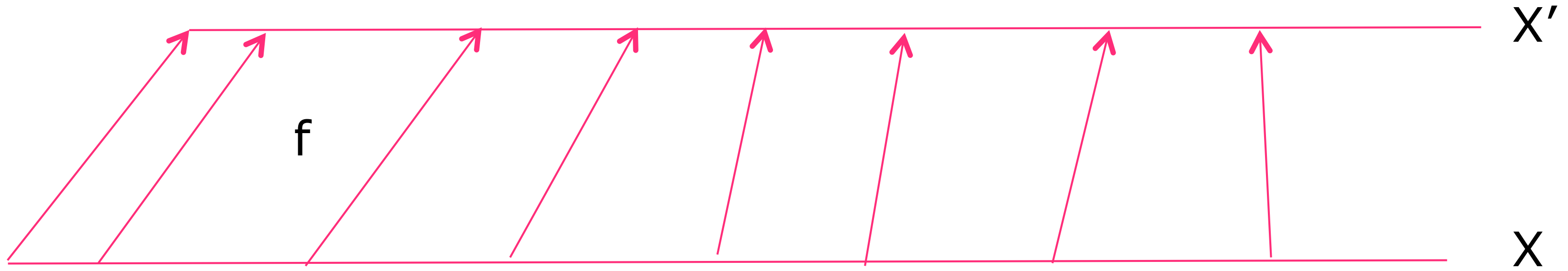


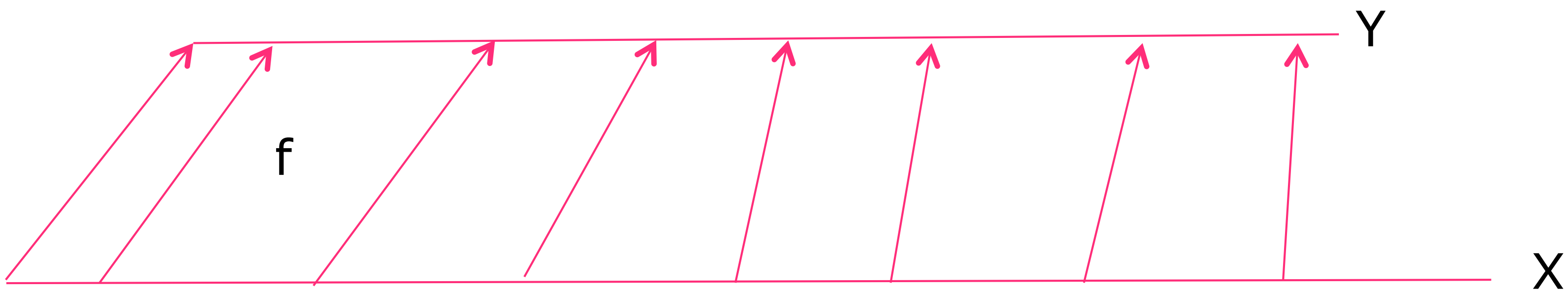
Order isomorphism

- (X,≤), (X′, ≤′) wosets
- f: X→X′ is an order isomorphism if
  - f is a bijection
  - $x<y \rightarrow f(x) <' f(y)$
- Write: $f:X \cong X'$

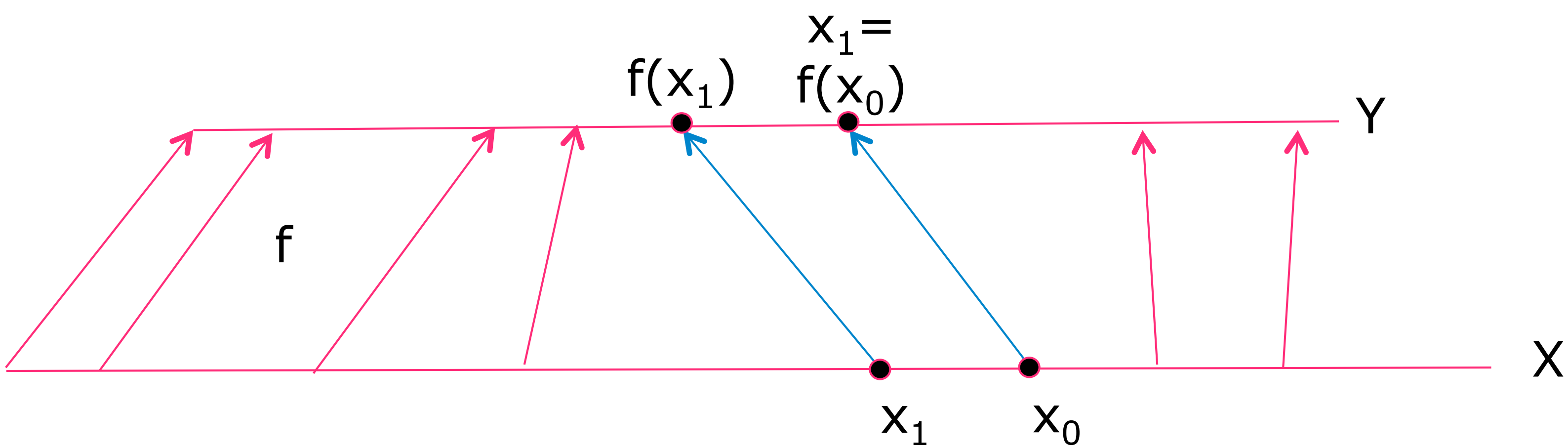


Important Theorem

- Let (X,≤) be a woset and Y⊆X.
- If f:X≅Y, then for all x: x ≤ f(x)



Proof



Let E = $\{x : f(x)<x\}$

Suppose x_0 = min(E). Consider $x_1 = f(x_0)$.

$x_1 < x_0$ → $f(x_1)<f(x_0)=x_1$ → $x_1 \in E$ → **Contradiction, as x_0 was minimal!**