Axiomatic set theory

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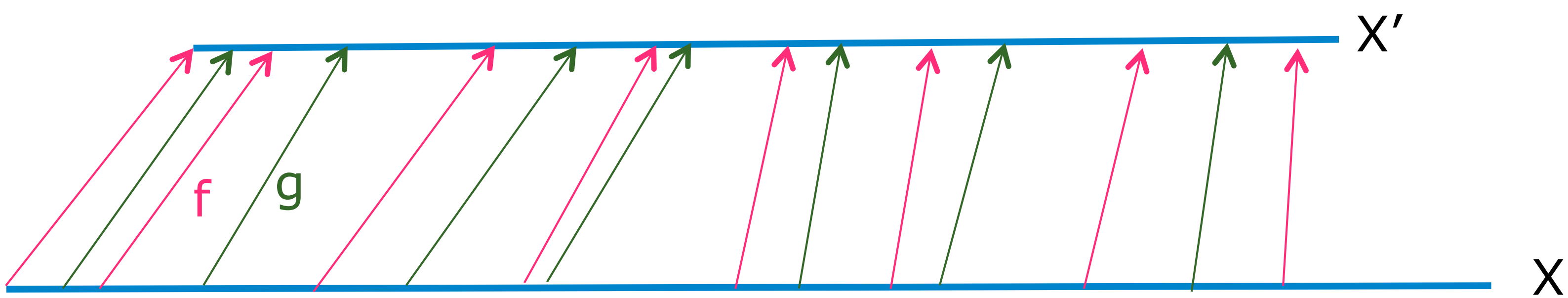
Important Theorem

- Let $(X,\leq)$ and $(X',\leq')$ be wosets.
- If $X \cong X'$, then there is exactly one

  $f{:}X \cong X'$.



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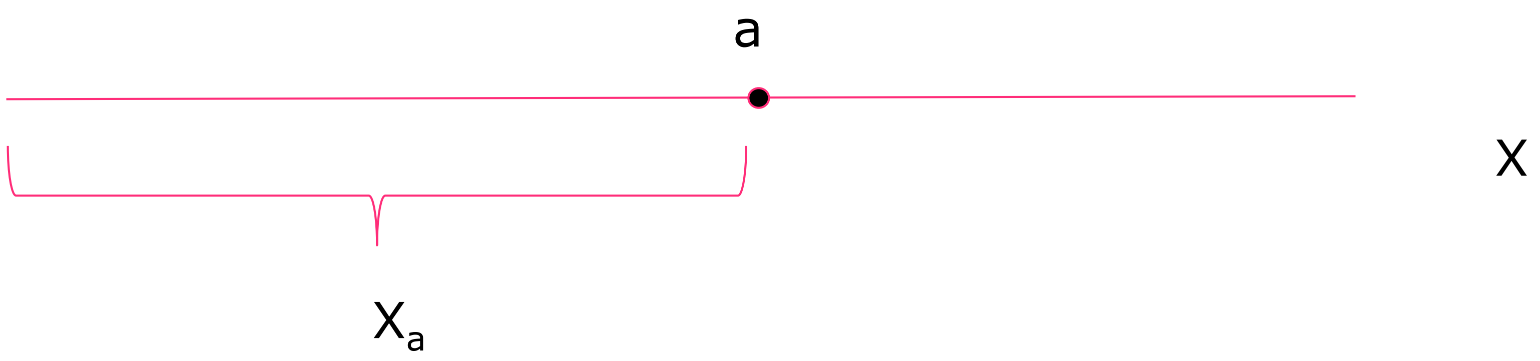
Proof

Let $h = f^{-1} \circ g$ → $h: X \cong X$ → $x \leq h(x)$ → $f(x) \leq g(x)$

Let $h' = g^{-1} \circ f$ → $h': X \cong X$ → $x \leq h'(x)$ → $g(x) \leq f(x)$

$$f(x) = g(x)$$

Segments



$X_a = \{x : x < a\}$

Theorem

- No woset is isomorphic to any of its own segments.
- Suppose $f:X \cong X_a$
- X_a is also a woset
- Hence $a \leq f(a)$
- But $f(a) \in X_a$, so $f(a) < a$, a contradiction.

Theorem:

- Let $(X,\leq)$ be a woset
- Let $A = \{X_a : a \in X\}$
- Then $(X,\leq) \cong (A, \subseteq)$

- Proof: Let $f(a)=X_a$. QED

Ordinals

- A woset $(X,\leq)$ is an ordinal if

$$X_a = a \text{ for all } a \in X.$$

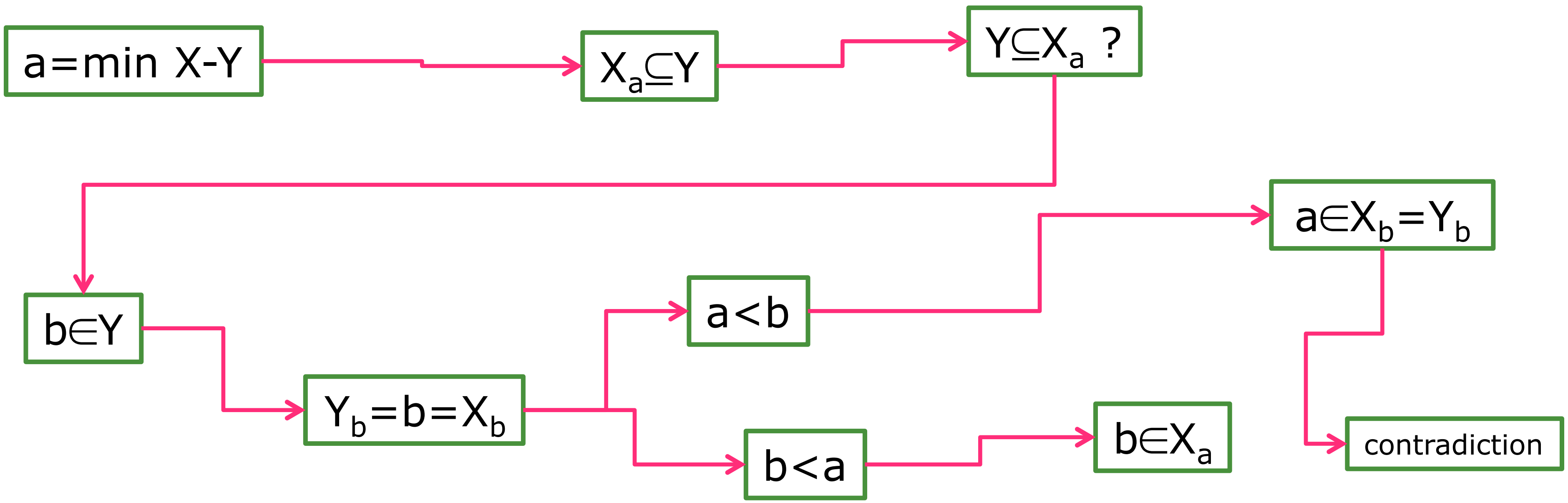
For $a,b \in X$, $a<b$ iff $X_a \subset X_b$ iff $a \subset b$ iff $a \in b$

Segments of ordinals are ordinals

- Suppose X is an ordinal.
- Suppose $a \in X$.
- We show X_a is an ordinal.
- Let $b \in X_a$.
- $(X_a)_b = \{x \in X_a : x < b\} = \{x \in X : x < b\} = b$.

Theorem

- Suppose X is an ordinal and $Y \subset X$.
- If Y is an ordinal, it is a segment of X.
- Proof:



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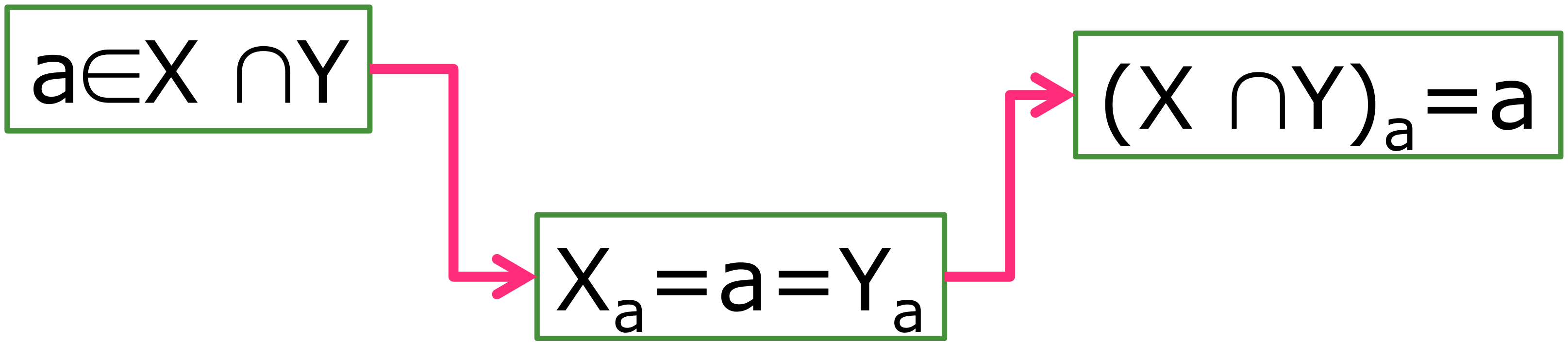
Intersection of ordinals is an ordinal

- X and Y ordinals
- Then $X \cap Y$ is an ordinal.



Theorem

- Let X and Y be distinct ordinals. Then one is a segment of the other.

Proof:

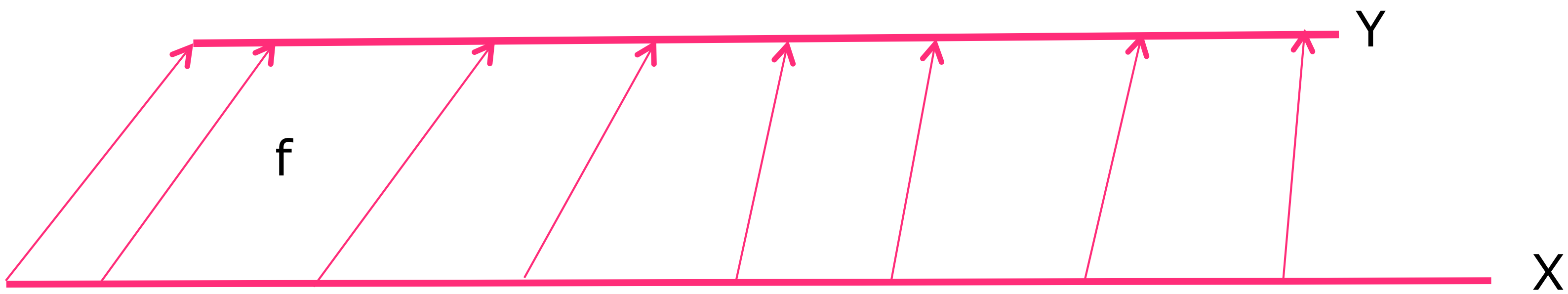
- Otherwise $X \cap Y \subset X$ and $X \cap Y \subset Y$.
- $X \cap Y$ is an ordinal.
- $X \cap Y = X_a = Y_b$
- $a = X_a = Y_b = b$
- $a = b \in X \cap Y$
- contradiction.

Isomorphic ordinals are identical

- Follows from the above, as a woset cannot be isomorphic to a proper initial segment of itself.

Important Theorem

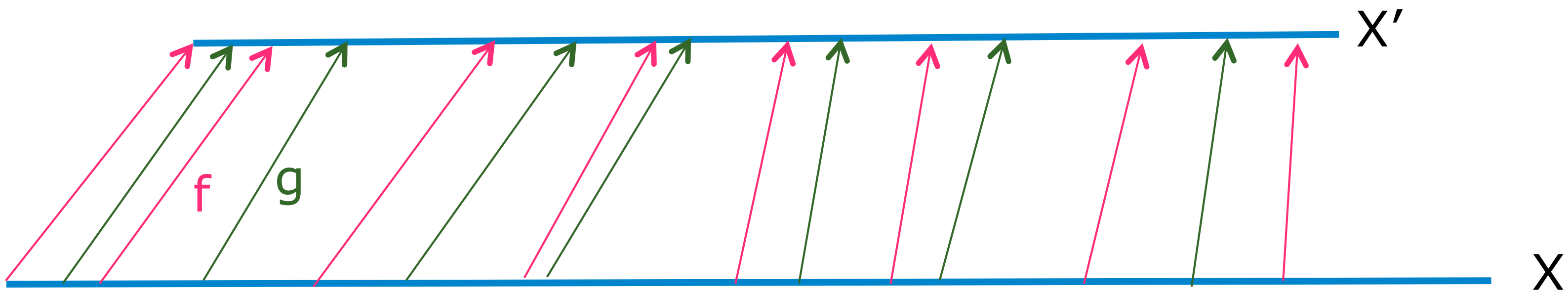
- Let $(X,\leq)$ be a woset and $Y\subseteq X$.
- If $f:X\cong Y$, then for all x: $x \leq f(x)$



Important Theorem

- Let $(X,\leq)$ and $(X',\leq')$ be wosets.
- If $X \cong X'$, then there is exactly one

$$f: X \cong X'.$$



Theorem

- No woset is isomorphic to any of its own segments.

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Ordinals

- A woset $(X,\leq)$ is an ordinal if

$$X_a = a \text{ for all } a \in X.$$

For $x,y \in X$, $x<y$ iff $X_x \subset X_y$ iff $x \subset y$

Segments of ordinals are ordinals

- Suppose X is an ordinal.
- Suppose $a \in X$.
- We show X_a is an ordinal.
- Let $b \in X_a$.
- $(X_a)_b = \{x \in X_a : x < b\} = \{x \in X : x < b\} = b$.

Intersection of ordinals is an ordinal

- X and Y ordinals
- Then $X \cap Y$ is an ordinal.

$$a \in X \cap Y \rightarrow X_a = a = Y_a \rightarrow (X \cap Y)_a = a$$

Theorem

- Let X and Y be distinct ordinals. Then one is a segment of the other.
- Otherwise $X \cap Y \subset X$ and $X \cap Y \subset Y$.
- $X \cap Y$ is an ordinal.
- contradiction.

Isomorphic ordinals are identical

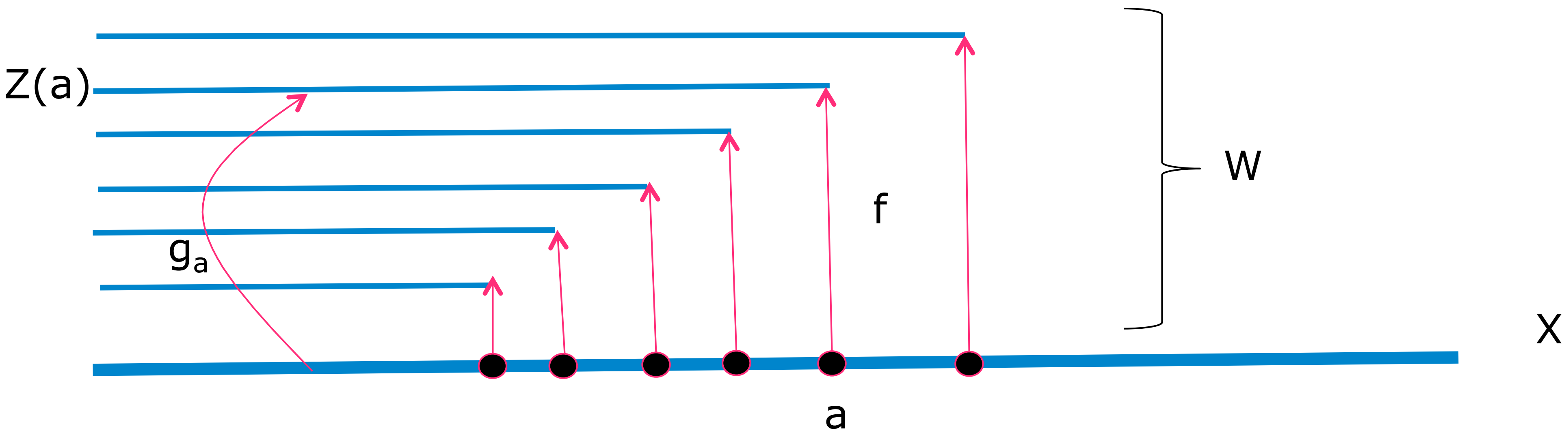
- Follows from the above.

Theorem

- A woset $(X,\leq)$.
- X_a is isomorphic to an ordinal for all $a\in X$
- Then X itself is isomorphic to an ordinal.

Proof

- $g_a : X_a \cong Z(a)$, $Z(a)$ ordinal
- g_a and $Z(a)$ are unique
- Let $W=\{Z(a) : a \in X\}$, $f(a)=Z(a)$



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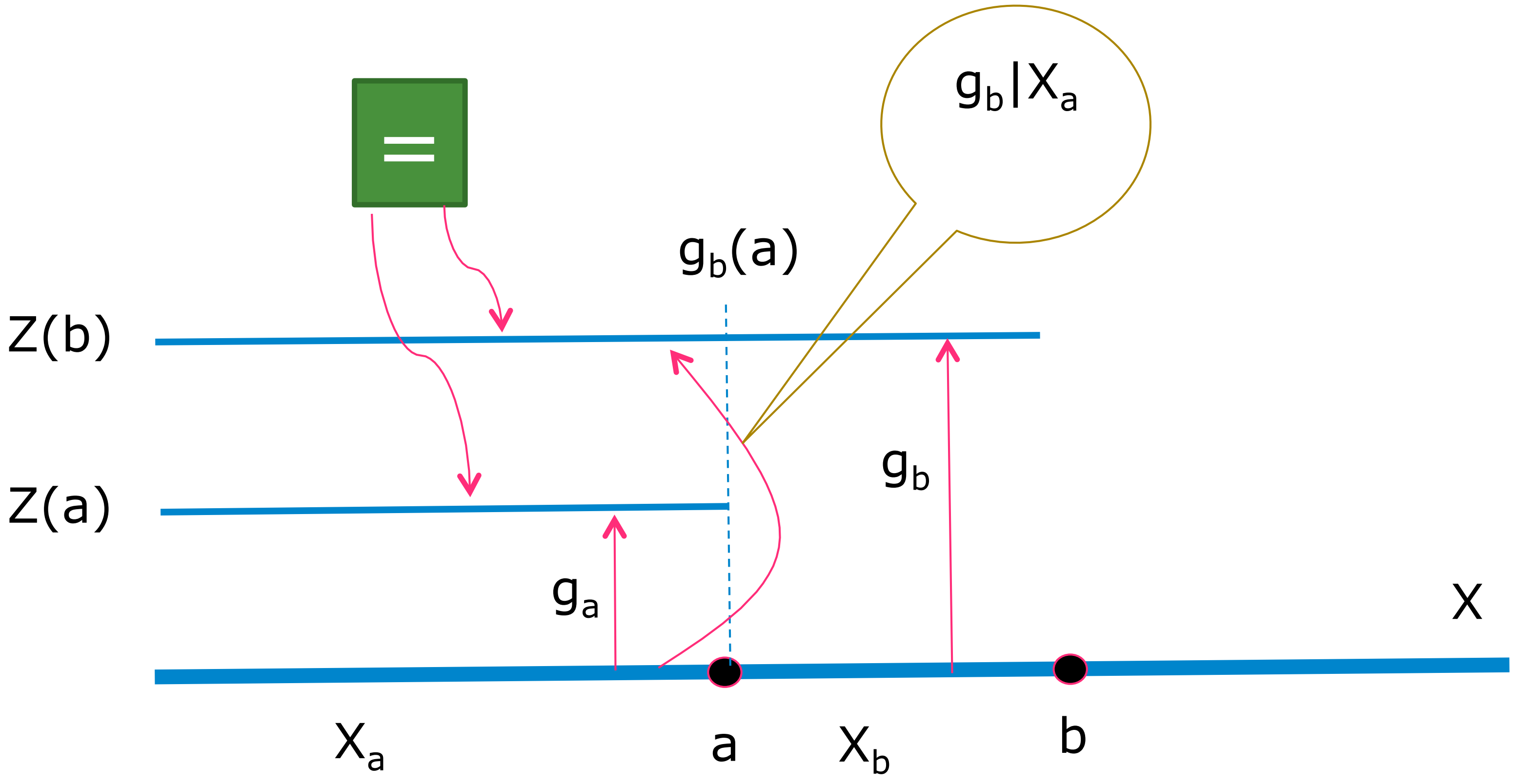
Claim: a<b implies f(a) ⊂ f(b)

- We show a < b implies Z(a) ⊂ Z(b)



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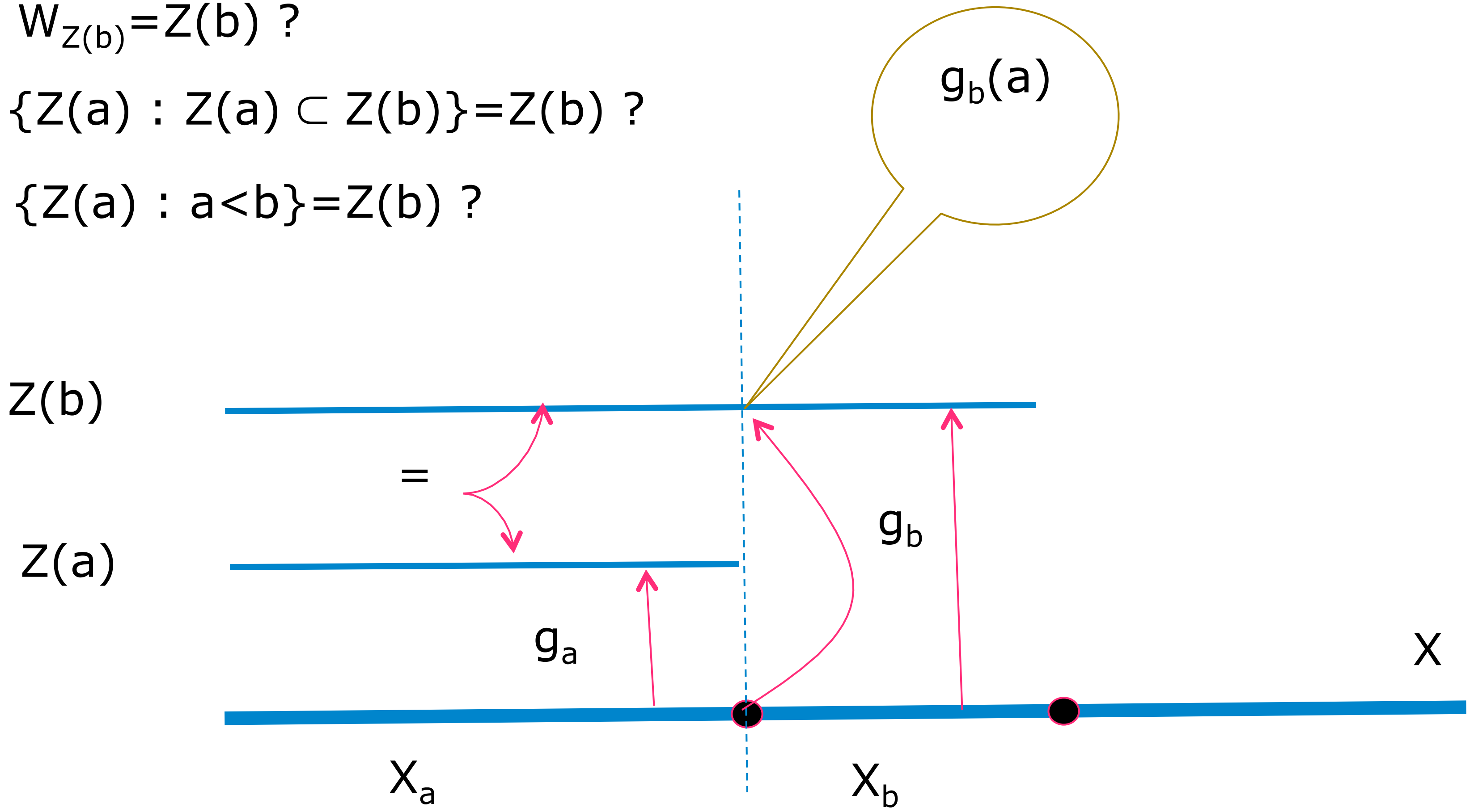
W is an ordinal i.e. $W_a = a$ for all $a \in W$

$W = \{Z(b) : b \in X\}$

$W_{Z(b)} = Z(b)$?

$\{Z(a) : Z(a) \subset Z(b)\} = Z(b)$?

$\{Z(a) : a < b\} = Z(b)$?



W is an ordinal i.e. $W_x = x$ for all $x \in W$

$W = \{Z(b) : b \in X\}$

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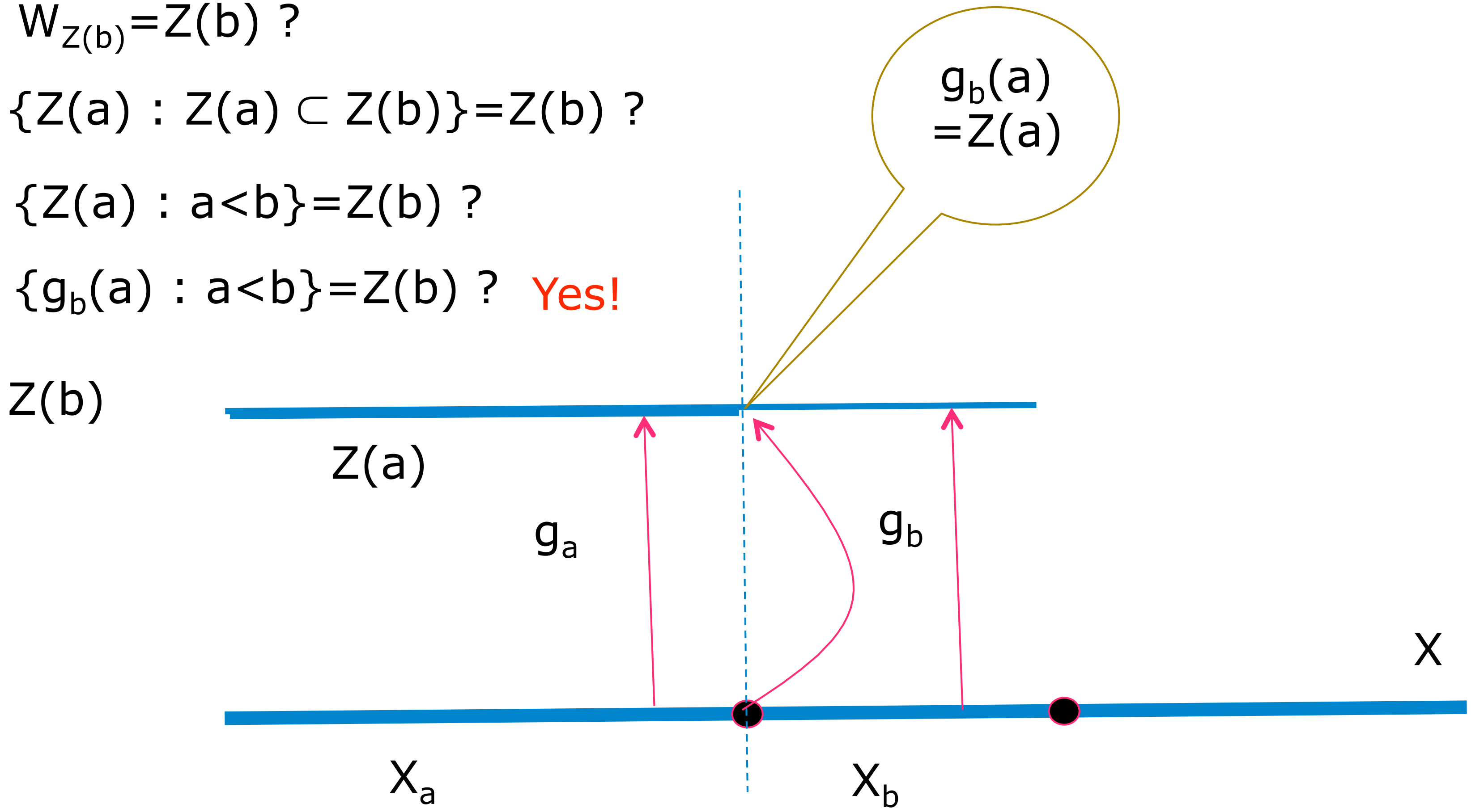
$\{Z(a) : Z(a) \subset Z(b)\} = Z(b)$?

$\{Z(a) : a < b\} = Z(b)$?

$\{g_b(a) : a < b\} = Z(b)$? Yes!



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