Axiomatic set theory

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Every woset is isomorphic to an ordinal

- Suppose $(X,\leq)$ is a woset.
- Let $E=\{a\in X : X_a$ is isomorphic to an ordinal$\}$.
- By Induction Principle and the previous theorem, $E=X$.
- By the previous theorem X is isomorphic to an ordinal.

The ordinal of a woset

- Every woset $(X,\leq)$ is isomorphic to a unique ordinal On(X)
- Ordinals are well-ordered by $\subset$
- α, β, γ, δ
- $\alpha=\{\beta : \beta<\alpha\}$

Ordinals

- $0=\emptyset$
- $1=\{0\}$
- $2=\{0,1\}$
- $3=\{0,1,2\}$
- …
- $n=\{0,1,\ldots,n-1\}$

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Infinite ordinals

- $\omega=\{0,1,2,3,\ldots,n,\ldots\}$
- $\{0,1,2,3,\ldots,n,\ldots,\omega\}=\omega\cup\{\omega\}$
- Denote: $\omega\cup\{\omega\}=\omega+1$
- More generally: $\alpha\cup\{\alpha\}=\alpha+1$
- Successor ordinals
- $\{0,1,2,\ldots,\omega,\omega+1,\ldots\}$ not a successor
- Limit ordinal

Transfinite sequences of sets

- A sequence is a function f such that dom(f) is an ordinal.
- $f = \langle x_\xi : \xi < \alpha \rangle$
- $f(\xi) = x_\xi$
- Special case $\langle x_n : n < \omega \rangle = \{x_n\}_{n=0}^{\infty}$

Paradoxes of naive set theory

- Burali Forti Paradox: The collection of all ordinals is wellordered by $\in$. If it were a set, it would be the largest ordinal. But there is no largest ordinal. So what goes wrong?
- Russell's Paradox: Let $R=\{x : x\notin x\}$. We know that $R\notin R$, because otherwise both $R\notin R$ and $R\in R$. Since $R\notin R$, we have $R\notin R$ and $R\in R$, a contradiction. What went wrong?

Reasons for axiomatization

- Paradoxes arise from confusion of concepts.
- By stating exact axioms and rules of derivation we can (try to) avoid confusions.
- We present so called Zermelo-Fraenkel axiomatization of set theory
- No paradoxes have been found in this.

Language of set theory

- We adopt the symbols:
  - $\in$ epsilon symbol
  - $=$ identity symbol
  -),(bracket symbols
  - v_n variable symbols
  - w_n constant symbols
  - $\neg$ negation symbol
  - $\vee$ disjunction symbol
  - $\wedge$ conjunction symbol
  - $\forall$ universal quantifier symbol
  - $\exists$ existential quantifier symbol

Formulas of set theory

- ($t \in t'$) where t is a variable symbol of a constant symbol
- ($t = t'$) where t is a variable symbol of a constant symbol
  - ($\neg\varphi$) negation of formula φ
  - ($\varphi \vee \psi$) disjunction of formulas φ and ψ
  - ($\varphi \wedge \psi$) conjunction of formulas φ and ψ
  - ($\forall x_n \varphi$) universally quantified formula
  - ($\exists x_n \varphi$) existentially quantified formula

Abbreviations

- $(\varphi \rightarrow \psi)$ abbreviates $((\neg\varphi)\vee\psi)$
- $(\varphi \leftrightarrow \psi)$ abbreviates $((\varphi\rightarrow\psi) \wedge(\psi\rightarrow\varphi))$
- $x\subseteq y$ $(\forall z((z\in x)\rightarrow(z\in y)))$
- $x\subset y$ $(x\subseteq y \wedge(\neg x=y))$
- $x=\{y\}$ $(\forall z((z\in x) \leftrightarrow (z=y)))$
- $x=\{y,v\}$ $(\forall z((z\in x) \leftrightarrow ((z=y)\vee(z=v)))$

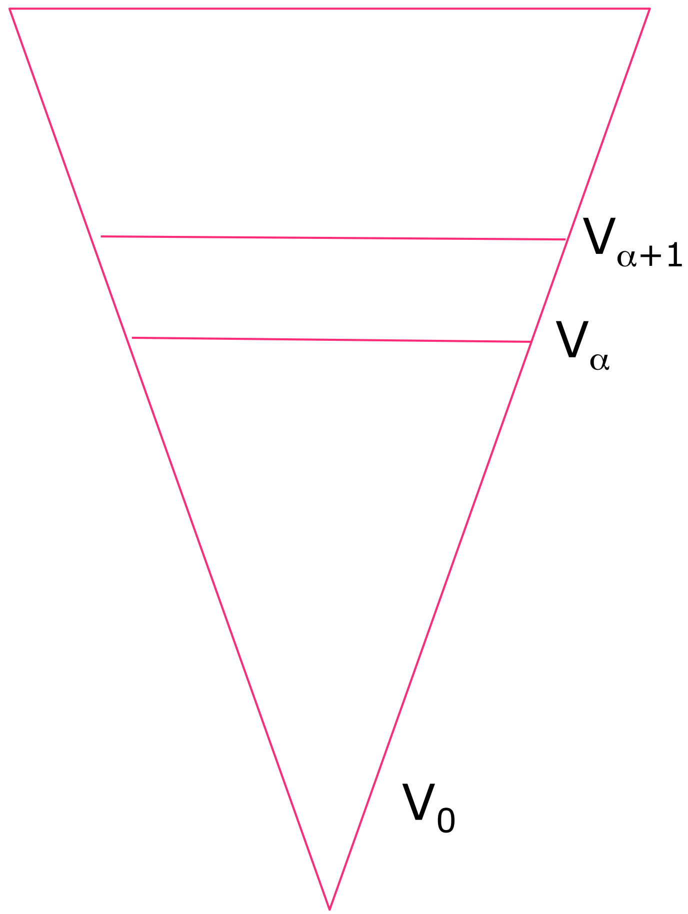
Last viewed

Further abbreviations

- Brackets are left out if it is obvious where they should be, so we write
- $\varphi \wedge \psi \wedge \theta$ instead of $((\varphi \wedge \psi) \wedge \theta)$
- Or instead of $(\varphi \wedge (\psi \wedge \theta))$

Intuitive universe of sets: The cumulative hierarchy

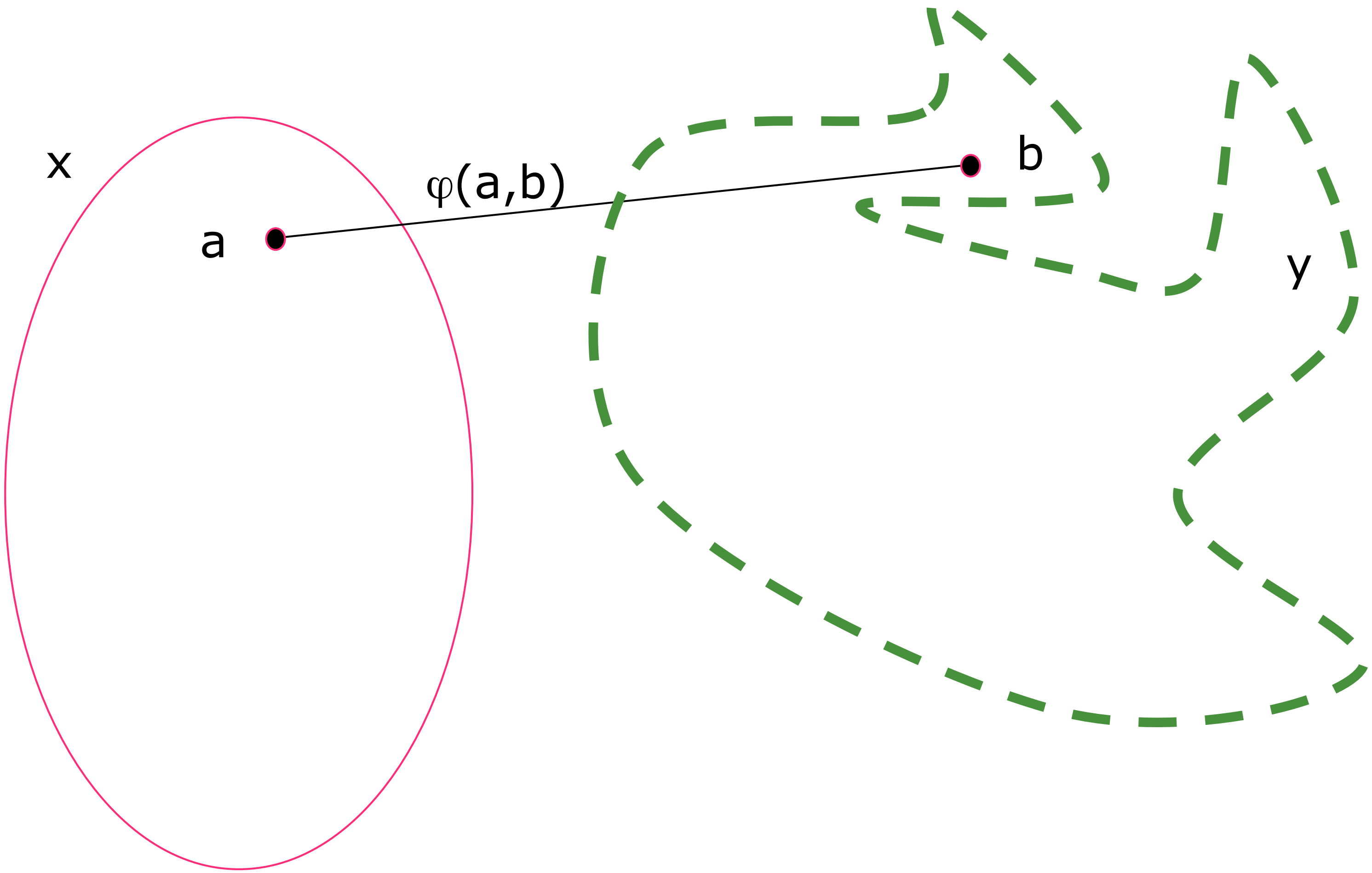
- $V_0 = \varnothing$
- $V_{\alpha+1} = P(V_\alpha)$
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for limit ordinals α



Zermelo-Fraenkel Axioms ZFC

1) Axiom of Extensionality: If two sets have the same elements, they are equal
2) Null Set Axiom: There is a set $\varnothing$ with no elements
3) Axiom of Infinity: There is a set x such that $\varnothing \in x$ and $\{a\} \in x$ whenever $a \in x$.
4) Power Set Axiom: If x is a set, there is a set P(x) consisting of all subsets of x.
5) Axiom of Union: If x is a set, there is a set $\cup x$ consisting of all elements of elements of x.
6) Axiom of Replacement: Let $\varphi(v_n,v_m)$ be a formula of the language of set theory such that for all a there is a unique b with $\varphi(a,b)$. Let x be a set. Then there is a set y consisting of exactly the elements b for which $\varphi(a,b)$ for some a in x.
7) Axiom of Subset Selection: Let $\varphi(v_n)$ be a formula of the language of set. Let x be a set. Then there is a set y consisting of exactly the elements a of x for which $\varphi(a)$ holds.
8) Axiom of Foundation: If x is a set, then there is a in x such that $a \cap x = \varnothing$.
9) Axiom of Choice: If x is a set of pairwise disjoint non-empty sets, then there is a set y which contains exactly one element from each set in x.

Axiom of Replacement



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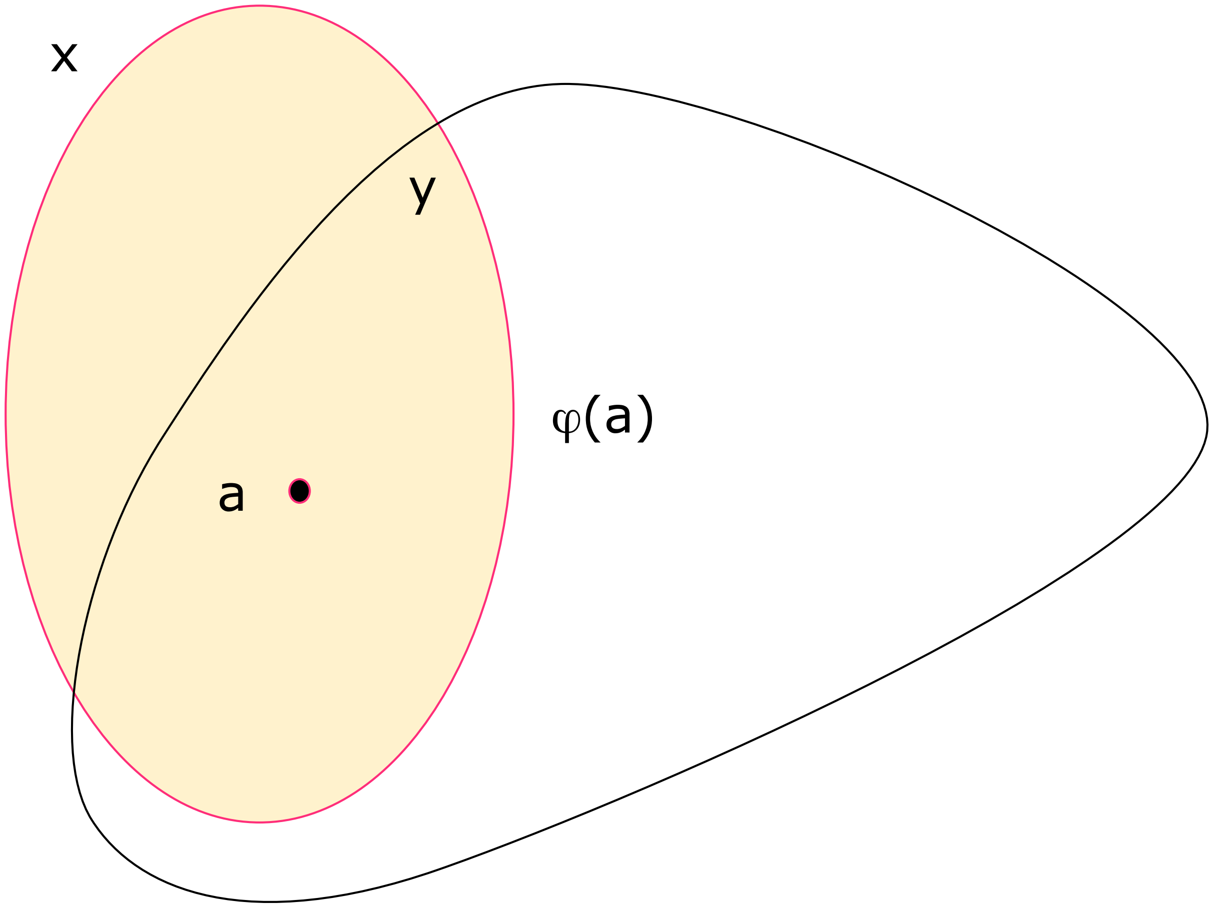
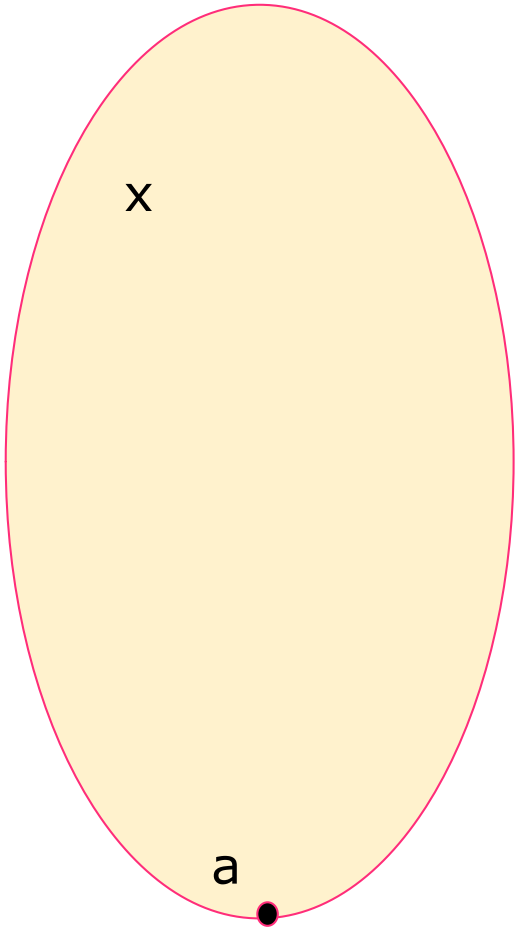
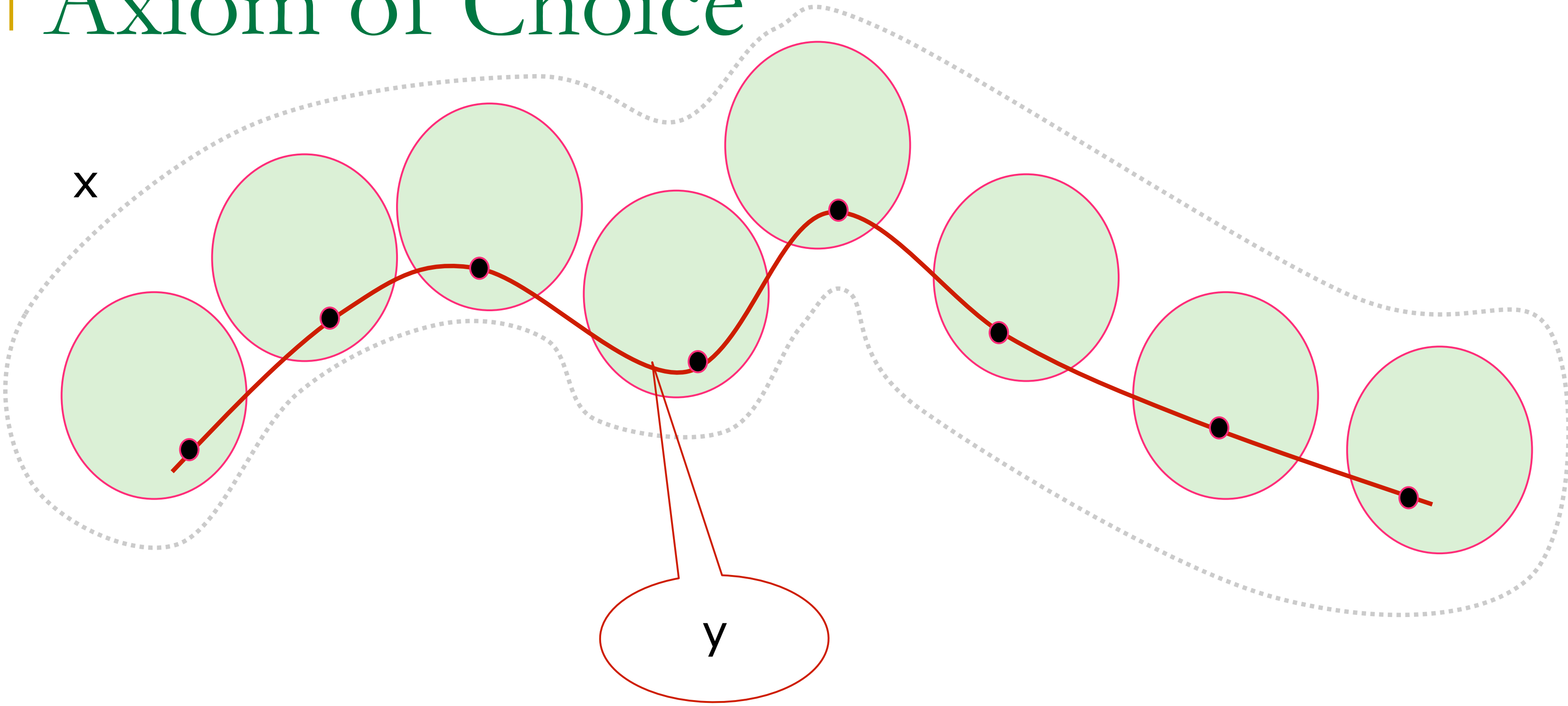
Axiom of Subset Selection



Axiom of Foundation



Axiom of Choice



Classes and sets

- Let $\varphi(v_n)$ be a formula of the language of set theory.
- $A=\{a : \varphi(a)\}$ is called a class.
- If $\varphi(a)$, then a is called an element of the class A, denoted $a\in A$.
- If a is a set, it is a class, $a=\{x : x\in a\}$.
- Some classes are not sets, e.g. $\{x:x\notin x\}$, $\{x : x=x\}$, $\{x:x$ is a singleton$\}$, etc . The are proper classes.

Operations on classes

- $A \cap B$
- $A \cup B$
- A-B
- A x B
- $R \subseteq A \times B$
- $F: A \rightarrow B$
- Injection, bijection

Construction of the cumulative hierarchy

- $V_0 = \emptyset$, exists by the Null Set Axiom.
- $V_{\alpha+1} = P(V_\alpha)$, exists by the Power Set Axiom.
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for limit ordinals α, exists: Let $A=\{V_\beta : \beta<\alpha\}$. By Axiom of Replacement A exists. Now $V_\alpha = \bigcup A$.