
Axiomatic set theory

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Misprints!

- Note that the textbook has misprints.
- Course www page has a link to the list.

Transfinite recursion

- Recursive definitions refer to themselves as in the Fibonacci sequence:
 - $f(0)=0$
 - $f(1)=1$
 - $f(n+2)=f(n)+f(n+1)$
- Note that these equations define **one and only one** function $f:\omega\rightarrow\omega$.

The goal of this lecture

- **Theorem:** Suppose $h: \text{On} \times V \rightarrow V$ is a function. Then there is a unique function $f: \text{On} \rightarrow V$ such that for all α

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha).$$

- Uniqueness is an important part of this theorem. The whole proof is based on it.
- Note that h and f are **classes**. That is one of the difficulties.

An easier version

- **Theorem:** Suppose $h: \text{On} \times V \rightarrow V$ is a function and λ is an ordinal. Then there is a unique function $f: \lambda \rightarrow V$ such that for all $\alpha < \lambda$:

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha).$$

- We first concentrate on the uniqueness part.

A little lemma about uniqueness

- Suppose $f:\mu\rightarrow V$ and $f':\mu\rightarrow V$ such that for all $\alpha<\mu$:

$$f(\alpha)=h(\alpha,f\upharpoonright\alpha) \text{ and } f'(\alpha)=h(\alpha,f'\upharpoonright\alpha).$$

- **Claim:** Then $f(\alpha)=f'(\alpha)$ for all $\alpha<\mu$.
- **Proof:** Suppose not! Let α be the least α such that $f(\alpha)\neq f'(\alpha)$. Thus $f(\beta)=f'(\beta)$ for all $\beta<\alpha$. In other words, $f\upharpoonright\alpha = f'\upharpoonright\alpha$. But then

$$f(\alpha)=h(\alpha,f\upharpoonright\alpha)=h(\alpha,f'\upharpoonright\alpha)=f'(\alpha),$$

contrary to assumption. QED

A further little note about uniqueness

- Suppose $f:\mu\rightarrow V$ and $f':\mu'\rightarrow V$, $\mu<\mu'$, such that for all $\alpha<\mu$: $f(\alpha)=h(\alpha,f\upharpoonright\alpha)$, and for all $\alpha<\mu'$: $f'(\alpha)=h(\alpha,f'\upharpoonright\alpha)$.
- **Claim:** Then $f=f'\upharpoonright\mu$.
- The claim is: $f(\alpha)=f'(\alpha)$ for all $\alpha<\mu$. Suppose not! Let α be the least α such that $f(\alpha)\neq f'(\alpha)$. Thus $f(\beta)=f'(\beta)$ for all $\beta<\alpha$. In other words, $f\upharpoonright\alpha=f'\upharpoonright\alpha$. But then

$$f(\alpha)=h(\alpha,f\upharpoonright\alpha)=h(\alpha,f'\upharpoonright\alpha)=f'(\alpha),$$

contrary to assumption. QED

Experiments

- Surely we can find some μ and $f:\mu\rightarrow V$ such that for all $\alpha<\mu$: $f(\alpha)=h(\alpha,f\upharpoonright\alpha)$.
- $\mu=0$ and $f=\emptyset$.
- $\mu=1$ and $f=\{(0,h(0,\emptyset))\}$.
- $\mu=2$ and $f=\{(0,h(0,\emptyset)), (1,h(1,\{(0,h(0,\emptyset))\}))\}$.
- Etc, etc

Now existence

- We note that the various $f:\mu\rightarrow V$ for various μ extend each other. Since such f is unique, let us denote it by f_μ .
- Uniqueness means that $\alpha<\beta \rightarrow f_\alpha\subseteq f_\beta$.
- But for how big μ can we form the function f_μ ?

An inductive argument of existence

- Let $E = \{\mu \leq \lambda : f_\mu \text{ exists}\}$.
- Claim: $E = \lambda + 1$.
- Proof: Otherwise there is some $\mu \leq \lambda$ such that μ is not in E . Let μ be the smallest such μ . So we have f_α for $\alpha < \mu$ and $\alpha < \beta \rightarrow f_\alpha \subseteq f_\beta$.
- By the Axiom of Replacement, the set $\{f_\alpha : \alpha < \mu\}$ exists. hence we can take its union:

$$f = \bigcup \{f_\alpha : \alpha < \mu\}$$

and the union is a function.

Two cases

- **Case 1:** μ is a limit ordinal, i.e. $\mu = \bigcup \mu$.

Now $f: \mu \rightarrow V$ and for all $\alpha < \mu$: $f(\alpha) = h(\alpha, f \upharpoonright \alpha)$.

So μ is in E , a contradiction.

- **Case 2:** $\mu = \beta + 1$. Now $f: \beta \rightarrow V$. Let $f' = f \cup \{(\beta, h(\beta, f))\}$.

We have $f': \mu \rightarrow V$ and for all $\alpha < \mu$:

$f'(\alpha) = h(\alpha, f' \upharpoonright \alpha)$. So again μ is in E , a contradiction.

- We must conclude that $E = \lambda + 1$.
- Hence λ is in E . QED

Back to the original goal:

- **Theorem:** Suppose $h: \text{On} \times V \rightarrow V$ is a function. Then there is a unique function $f: \text{On} \rightarrow V$ such that for all α

$$f(\alpha) = h(\alpha, f \upharpoonright \alpha).$$

Note: The previous proof only gave such $f: \lambda \rightarrow V$ for any given λ .

Proof

- We define $f(x)=y$ by the formula:
“There is an ordinal λ and a function $g:$
 $\lambda \rightarrow V$ such that $x \in \lambda$ and for all $\alpha < \lambda:$
 $g(\alpha) = h(\alpha, g \upharpoonright \alpha)$, and $y = g(x)$.”