CAPITA SELECTA: MODEL THEORY, AXIOMATIC SET THEORY

TOPIC FOR 2009: DESCRIPTIVE SET THEORY

ASSIGNMENT 3 (DUE 26 OCTOBER 2009)

(a) Show that the class of Borel sets is closed under continuous inverse images.

(b) Show that for each $n \in \mathbb{N}$, $i = 0, 1$ we have $\Sigma^i_n \not\subseteq \boldsymbol{\Pi^i_n}$ and $\Pi^i_n \not\subseteq \boldsymbol{\Sigma^i_n}$.

(c) Show that a closed set in $\mathcal{N} \times \mathcal{N}$ is the set of infinite paths through a tree on $\omega \times \omega$.

(d) Let T be a tree on $\omega \times \omega$. Also let T^α be the α-column of this two dimentional tree, i.e.

$$T^\alpha = \{s \mid \langle \alpha \upharpoonright |s|, s\rangle \in T\}.$$

Show that the function $\alpha \to T^\alpha$ is continuous (here we identify the tree T^α with its code).

(e) Let $\mathcal{X}, \mathcal{Y}$ be Polish spaces (complete, metrizable and separable). Show that if the graph of a function $f : \mathcal{X} \to \mathcal{Y}$ ($Graph(f) = \{(x, y) \mid f(x) = y\}$) is $\boldsymbol{\Sigma}^1_1$ then f is Borel.
Hint: $f(x) \in N_s \iff \exists y\ [f(x) = y \ \wedge \ y \in N_s] \iff \forall y[f(x) = y \Rightarrow y \in N_s]$.

(f) Suppose $f : \mathcal{X} \to \mathbb{R}$ is a Borel function and for each $x \in \mathcal{X}$ there is exactly one solution y of the equation

$$f(x, y) = 0$$

so that this equation determines y as a function of x, $y = g(x)$. Prove that g is a Borel function. In particular, if $f : \mathcal{X} \to \mathcal{Y}$ is a Borel bijection, then f^{-1} is also Borel, so f is a Borel isomorphism.
Hint: $g(x) = y \Rightarrow f(x, y) = 0$.

(g) Prove that for each Δ^1_1 pointset of the Baire space there is a computable function $\pi : \mathcal{N} \to \mathbb{N}$ and a Π^0_1 set $A \subseteq \mathcal{N}$ such that π is 1–1 on A and $\pi[A] = P$.
Hint: We have done the analog of this for $\boldsymbol{\Delta}^1_1$ (i.e. Borel). Emulate that proof, ensuring that now certain things are computable rather than merely continuous. Notions will be replaced with their 'effective' (i.e. computable) counterparts. For example, closed sets will become Π^0_1 sets (also called *effectively closed sets*).

(h) The class of computable functions can be defined as the smallest class containing some basic functions (zero, successor, projections) and closed under composition primitive recursion (if g is in the class then f defined by $f(n+1) = g(f(n), n)$ is in the class) and the so–called *least*-operator (if g is in the class then $f(n) = \mathit{least}\ t\ [g(n, t) = 0]$ is in the class).
Show that all (total) computable functions are Δ^0_1.
Hint: Use induction on the length of the expression of the function. After all a function is computable, if there is a finite sequence of derivation where we get it from the basic function by a finite application of the schemes (composition and 'least' operator). For the definition in the induction step, emulate the proof that all Borel sets are Δ^1_1 in a real parameter.

(i) A set P is called thin if it contains no perfect subsets. Show that P is thin iff every Borel subset of P is countable. Infer that the notion of being thin is preserved by Borel isomorphisms.
Hint: Use Cantor-Bendixon theorem.