CAPITA SELECTA: MODEL THEORY, AXIOMATIC SET THEORY

TOPIC FOR 2009: DESCRIPTIVE SET THEORY

ASSIGNMENT 4 (DUE 9TH OF NOVEMBER 2009)

(a) Show that a function is Δ^0_1 iff it is computable. *Hint:* From a Σ^0_1 and a Π^0_1 definition of f you can define f recursively. The other direction is similar (but simpler) to the theorem that all Borel sets are $\boldsymbol{\Delta}^1_1$.

(b) Show that the inverse image of a Π^0_1 set through a computably coded continuous function is Π^0_1. *Hint:* Note that the intersection of a computable sequence of Π^0_1 classes (as the sets of paths through computable trees) is Π^0_1. So prove that this inverse image is the intersection of the classes of paths through a computable sequence of trees. You could also work with closed sets, instead of using trees.

(c) Show that there are uncountably many well founded trees.

(d) Show that the height/length of a well founded tree is a countable ordinal.

(e) Show that given a countable ordinal σ, there are countably many well founded trees of height σ. So WF_σ is countable. *Hint:* By induction on σ.

(f) Prove the normal form theorem for Π^1_1 predicates in the space ω. That is, for every Π^1_1 predicate $P \subseteq \omega$ there is a computable sequence of (computable) trees T_n such that

$$P(n) \iff T_n \in WF \iff \langle T_n \rangle \in WFG. \tag{1}$$

Also prove the corresponding normal form for $\boldsymbol{\Pi}^1_1$ (as above, only without 'computable'). *Hint:* Suppose that P is Π^1_1. Then

$$n \in A \iff \forall \beta \exists m P(\beta, m, n)$$

for a computable predicate P. Without loss of generality we can assume that if $P(\beta, m, n)$ then $P(\beta, i, n)$ for all $i \geq m$. We need to make a suitable sequence of trees T_n as above, using P as ingredient. We just need to 'read-off' P as a tree. We let T_n be the set of $\beta \restriction m$ for all β and m such that $\neg P(\beta, m, n)$. Verify that this set of string is a (computable) tree, for each n. Then verify (1). A similar argument works for $\boldsymbol{\Pi}^1_1$.

(g) Let

$$\delta^1_1 = \sup\{||T|| \mid T \in WF \text{ and } T \text{ is } \Delta^1_1\}.$$

Show that $\delta^1_1 = \omega_1^{CK}$.
Hint: It suffices (why?) to show that

$$\delta^1_1 \leq \sup\{||T|| \mid T \in WF \text{ and } T \text{ is computable}\}.$$

Assume for a contradiction that some well-founded Δ^1_1 tree F has height which is greater than all computable ordinals. By the Π^1_1 representation theorem (or, normal form theorem) every Π^1_1 relation P can be written as

$$P(n) \iff T^n \in WF$$

where T^n is a computable function from ω to the space of computable trees. But then we have

$$P(n) \iff T^n \in WF \ \wedge \ ||T^n|| \leq ||F||.$$

Explain that this gives a Σ^1_1 definition of P. The result follows since $\Pi^1_1 \not\subseteq \Sigma^1_1$.

(h) Show that the class Δ^1_1 is the smallest class which contains the computable sets and is closed under effective unions and complements. *Hint:* First show that Δ^1_1 has the mentioned properties. Then by induction on the computable ordinals show that every class $\mathcal{M}$ with these closure properties contains Δ^1_1. Instead of the Σ, Π classes of the hyperarithmetic hierarchy you can use the fact that Δ^1_1 contains the sets which have computable Borel codes. Show that for every computable ordinal σ the sets which have Borel code α with $||T_\alpha|| = \sigma$ belong to $\mathcal{M}$.

(i) Show that every Δ^1_1 set is the image of a Π^0_1 set through a 1–1 computably coded continuous function from $\mathcal{N}$ to $\mathcal{N}$. *Hint:* Follow the corresponding boldface proof and replace the notions with their effective counterparts.