CAPITA SELECTA: MODEL THEORY, AXIOMATIC SET THEORY

TOPIC FOR 2009: DESCRIPTIVE SET THEORY

ASSIGNMENT 5 (DUE 23TH OF NOVEMBER 2009)

(a) Show that the projective functions $f_i : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$, $f_0(x, y) = x$, $f_1(x, y) = y$ are continuous.

(b) Show that the set of points in the Baire space that are onto as sequences of natural numbers form a $\mathbf{\Pi}^0_2$ complete set.

(c) Show that the set of points in the Baire space that tend to ∞ as sequences of natural numbers form a $\mathbf{\Pi}^0_3$ complete set.

(d) Show that $f : X \to Y$ is Borel iff $f^{-1}(O)$ is Borel for every open set of Y iff $f^{-1}(B)$ is Borel for every Borel set of Y iff $f^{-1}(N_s)$ is Borel for each s iff $\{(x, s) \mid x \in f^{-1}(N_s)\}$ is Borel in $X \times \mathbb{N}$.

(e) What is the highest cardinality that a Polish space can have? What is the highest cardinality that the space of continuous functions between two Polish spaces can have? What is the highest cardinality that the space of functions between two Polish spaces can have?

(f) Show that if X is a Polish space and (A_n) is a pairwise disjoint sequence of analytic sets then there are pairwise disjoint Borel sets (B_n) such that $A_n \subseteq B_n$.

(g) Show that the Borel sets in a Polish space are exactly the injective images by continuous (equivalently Borel) functions of the closed subsets of the Baire space $\mathcal{N}$.

(h) Show that under Zermelo-Frankel set theory, the axiom of choice is equivalent to the determinacy of closed games.

(i) Show the determinacy of closed games in the Baire space without using the axiom of choice.

(j) Give an example of a set in the Baire space that is not determined. *Hint:* Take the set we constructed in order to show that not all set have the perfect set property. Explain why this is not determined.

(k) **The Axiom of Dependent Choices** states that any nonempty pruned tree on a set A has an infinite branch. Show that the Axiom of Dependent Choices is equivalent to the statement that in all games $G(T, X)$ it is not possible that both players have winning quasistrategies.

(l) Show that If Γ is one of the Σ/Π pointclasses and has the reduction property, then it does not have the separation property. In particular, $\mathbf{\Pi}^1_1$ does not have the separation property. *Hint:* Let $G \subseteq \mathbb{N} \times \mathbb{N}$ be universal for $\Gamma \restriction \mathbb{N}$ and let

$$P(\langle n_0, n_1\rangle) \iff (n_0, \langle n_0, n_1\rangle) \in G \qquad Q(n) \iff (n_1, \langle n_0, n_1\rangle) \in G.$$

Choose P^*, Q^* in Γ which reduces P, Q and for a contradiction assume that some R separates P^* from Q^*. Now choose e, m such that

$$R(n) \iff (e, n) \in G \qquad \neg R(n) \iff (m, n) \in G$$

and let $t = \langle m, e\rangle$. Show that both assumptions $t \in R$, $t \notin R$ lead to contradictions.