CAPITA SELECTA: MODEL THEORY, AXIOMATIC SET THEORY

TOPIC FOR 2009: DESCRIPTIVE SET THEORY

ASSIGNMENT 1 (DUE 12 OCTOBER 2009)

(a) Show that there are uncountably many open sets in the Cantor space.

(b) Show that there are uncountably many clopen sets in the Baire space.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function on the real line. Prove that the set

$$A = \{x \in \mathbb{R} \mid f \text{ is continuous at } x\}$$

is G_δ (i.e. $\mathbf{\Pi}^0_2$).

Hint: Define the variation of f on an interval (a, b) by

$$V(a,b) = \sup\{f(x) \mid x \in (a,b)\} - \inf\{f(x) \mid x \in (a,b)\}.$$

The local variation of f is given by

$$v(x) = \lim_{n\to\infty} V(x - \frac{1}{n}, x + \frac{1}{n})$$

and it is not hard to show that f is continuous at x exactly if $v(x) = 0$. Let $A_n = \{x \mid v(x) < 1/n\}$. Show that A_n is open and $A = \cap_n A_n$.

(d) We say that a class of $\mathcal{D}$ of relations is closed under continuous substitution if, whenever $P \in \mathcal{D}$ (and $P \subseteq \mathcal{Y}$ for some space $\mathcal{Y}$), and $f : \mathcal{X} \to \mathcal{Y}$ is a continuous function, the relation $f^{-1}[P]$ is also in $\mathcal{D}$. Recall that

$$x \in f^{-1}[P] \iff f(x) \in P \iff P(f(x)).$$

Show that the classes $\mathbf{\Sigma}^0_n, \mathbf{\Pi}^0_n, \mathbf{\Delta}^0_n$ are closed under continuous substitution, $\vee$ and $\wedge$.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on the real line. Prove that the relations

$$\begin{aligned} P(x,y) &\iff f'(x) = y \\ Q(x) &\iff f'(x) \text{ exists} \end{aligned}$$

are both $\mathbf{\Pi}^0_3$.

(f) Prove that if $P \subseteq \mathcal{X}$ and $Q \subseteq \mathcal{Y}$ are $\mathbf{\Sigma}^0_n$, then the product $P \times Q \subseteq \mathcal{X} \times \mathcal{Y}$ is also $\mathbf{\Sigma}^0_n$ (*Hint:* Use closure under continuous substitution).

(g) The definition of the transfinite Borel hierarchy could be given by transfinite induction on all ordinals (not only the countable ones). So for all ordinals κ, the class $\mathbf{\Sigma}^0_\kappa$ consists of the countable unions of sets in the previous classes $\mathbf{\Sigma}^0_\rho$ for $\rho < \kappa$. Show that in that case, the hierarchy collapses at level ω_1. In other words, that $\mathbf{B} = \mathbf{\Sigma}^0_\sigma$ for all $\sigma \geq \omega_1$.