Random reals and Lipschitz continuity

Length-efficient oracle computations and measures of relative randomness.

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The *sw* reducibility

The strong weak truth table reducibility was proposed as a tool for measuring relative randomness:

Definition

(Downey, Hirschfeldt, LaForte 2001)

- We say $A \leq_{sw} B$ if there is a Turing functional Γ and a constant c such that $\Gamma^B = A$ and the use of this computation on any argument n is bounded by $n + c$.
- The induced degree structure is the *sw* degrees.

Intuition

This reducibility

- formalizes the notion of length-efficient oracle computation
- $\alpha \leq_{sw} \beta$ means that for all n, $\alpha \upharpoonright n$ can be computed from $\beta \upharpoonright (n + c)$
- for $c = 0$ we get something more primitive: the identity bounded Turing reducibility
- occurs naturally in constructions in classical computability theory
- has other applications (Soare, Nabutovsky, Weinberger)

Partial computable operators

Definition

A partial operator Γ from a (pseudo) metric space (X, d) to itself is Lipschitz continuous if there is a constant C such that

$$d(\Gamma(x), \Gamma(y)) \leq C \cdot d(x, y) \tag{1}$$

for all x, y in the domain of Γ.

Partial computable operators

- Consider Turing functionals as functions $(2^\omega, d) \mapsto (2^\omega, d)$
- The metric d on 2^ω such that $d(\alpha, \beta) = 2^{-n}$ where n is the least position where α, β differ

An immediate approach for an effective version of Lipschitz continuity:

Definition

Let us say α is computably Lipschitz-0 reducible to β ($\alpha \leq_{\mathrm{cl}_0} \beta$) if there exists a Turing functional Γ such that $\Gamma^\beta = \alpha$ and which is Lipschitz continuous when considered as a map $(2^\omega, d) \mapsto (2^\omega, d)$.

Partial computable operators

- Γ is forced to behave well on all those reals α in its domain
- but the definition above makes no provision for the way in Γ will behave on finite initial segments which have no infinite extension in the domain.

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- but the definition above makes no provision for the way in Γ will behave on finite initial segments which have no infinite extension in the domain.

So the reducibility $\leq_{\mathrm{cl}_0}$ does not preserve randomness. In fact:

Proposition

On the Δ_2 degrees $\leq_{cl_0}$ and $\leq_T$ coincide.

Partial computable operators

A more fruitful approach is to consider Turing functionals as functions $(2^{<\omega}, d) \mapsto (2^{<\omega}, d)$.

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Partial computable operators

Proposition

An sw functional is a partial computable and Lipschitz continuous operator from $(2^{<\omega}, d)$ to itself. Conversely, every partial computable and Lipschitz continuous operator $\Gamma : (2^{<\omega}, d) \rightarrow (2^{<\omega}, d)$ equals an sw functional on infinite strings.

In other words, $\alpha \leq_{\mathrm{sw}} \beta$ iff $\alpha \leq_{\mathrm{cl}} \beta$.

Basic Notions

We use the following equivalent notions of randomness:

- Prefix-free complexity: incompressibility paradigm
- Martin-Löf tests: avoiding effectively null sets

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Of particular interest in computable analysis is the class of computably enumerable reals.

Definition

A real number is *computably enumerable (c.e.)* if it is the limit of a computable increasing sequence of rationals.

cl and other measures

- This was only one of the proposed measures
- other proposals like the *rK*-reducibility have been studied more
- *cl* reducibility has advantages over other measures like the Solovay reducibility
- The lack of join is considered a serious disadvantage
- $\leq_{cl}$ and Solovay reducibility coincide on the c.e. sets

cl and relative randomness

Proposition

(Downey, Hirschfeldt, LaForte) If $a \leq_{cl} b$ *are any reals then for all n, the prefix-free complexity of* $a \restriction n$ *is less than or equal to that of* $b \restriction n$ *(plus a constant). The same holds for plain Kolmogorov complexity.*

So

- $\leq_{cl}$ arguably qualifies as a *measure of relative randomness*
- and in particular it preserves randomness.

Yu-Ding Theorem

Theorem

(Yu, Ding 2004) There is no cl-complete c.e. real.

Corollary

(Downey, Hirschfeldt, LaForte)
The structure of cl-degrees is not an upper semi-lattice.

Enumerability, Randomness and $\leq_{cl}$

Are there c.e. reals above all c.e. sets?

Proposition

Every random c.e. real is cl-above every set in the finite levels of the difference hierarchy.

Enumerability, Randomness and $\leq_{cl}$

But are there non-random c.e. reals with this property?

Proposition

There are non-random c.e. reals cl-above every set in the finite levels of the difference hierarchy.

E.g. $a = \sum_{e\in\mathbb{N}} \sum_{n\in W_e} 2^{-(e+n+2)}$ is non-random and *cl*-above all c.e. sets.

Enumerability, Randomness and $\leq_{cl}$

Theorem

(B.) There are no cl-maximal c.e. sets. That is, for every c.e. set A, there exists a c.e. set W such that $A <_{cl} W$.

Comments

- Two proofs for this result
- Both of them are non-uniform: we uniformly construct sets W_1, W_2, W_3, one of which succeeds
- One uses
  - a positive enumeration assumption on the given set to construct W_1
  - a negative enumeration assumption on the given set to construct W_2, W_3
- The other uses three versions of a mixed assumption to uniformly construct W_1, W_2, W_3

Solovay Degrees

Corollary

(B.) The substructure of the Solovay degrees consisting of the ones with c.e. members (i.e. containing c.e. sets) has no maximal elements.

Computing with Random oracles

- If α is a random c.e. real then every c.e. real is Solovay computable from α.
- If α is a random c.e. real then every c.e. real is weak truth table computable from α.
- (Kučera) Every set is Turing computable from a random set.

Random oracles in the c.e. reals

Theorem

(B. and A.E.M. Lewis) There are c.e. reals α that cannot be cl-computed by any random c.e. real. That is, for any c.e. $\beta \geq_{cl} \alpha$, β is not random.

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- Two different proofs for this result
- (Hirschfeldt) There is a set which cannot be *cl*-computed by any random set

Theory of the *cl*-degrees

Theorem

(B. and A.E.M. Lewis) The existential theory of the c.e. cl-degrees is decidable. In particular, every finite partial order can be embedded in $(\mathcal{D}_{cl}, <)$, $(\mathcal{R}_{cl}, <)$.

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- The result does not follow by simply applying the usual proof for classical structures like the Turing or the wtt degrees: we have no join
- What about the decidability of higher levels of the theory?

Quasi-maximal *cl*-degrees

Theorem

(B. and A.E.M. Lewis) There is a quasi-maximal cl-degree $\mathbf{a}$*: any* $B \geq_{cl} A$ *is Turing computable from* A*. This can be below* $\mathbf{0}'$ *and non-random.*

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Theorem

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Corollary

(B. and A.E.M. Lewis) There exist low reals which are of quasi-maximal cl *degree.*

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Corollary

(B. and A.E.M. Lewis) The cl *degrees are not an upper semi-lattice, in fact there exist two* cl *degrees with no upper bound.*

Theorem

(B. and A.E.M. Lewis) No random real is of maximal cl *degree.*

Questions

- Are there (globally) maximal *cl*-degrees?
- Are there locally (e.g. in the c.e. reals) maximal *cl*-degrees?
- Are there many degrees of random c.e. reals?

End

Thank you!