Kolmogorov complexity and c.e. sets

Questions and directions

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The K-degrees

K denotes the prefix-free complexity.

$X \leq_K Y$ if $\exists c \forall n\, K(X \restriction_n) \leq K(Y \restriction_n) + c$.

The induced degree structure is known as the K-degrees.

$X \leq_{LK} Y$ if $\exists c \forall \sigma\, K^Y(\sigma) \leq K^X(\sigma) + c$.

The induced degree structure is known as the LK-degrees.

Minimal pairs in the K-degrees

- Csima and Montalbán: Existence (in Δ^0_4);
- Merkle and Stephan: Existence in Σ^0_2;
- Barmpalias and Vlek: A Σ^0_2 capping each Σ^0_1 to **0**.

Is there a minimal pair of c.e. reals in the K-degrees?

...a question of Downey and Hirschfeldt

Complete sets of very low complexity

Theorem

Let A be a c.e. set which is not K-trivial. Then there exists a computably enumerable set B such that $B \equiv_T \emptyset'$ and $B <_K A$.

...complete sets can have arbitrarily low nontrivial complexity.

And a uniform version:

Theorem

Let A, D be c.e. and not K-trivial. Then there exists a c.e. set B such that $B <_K A$, $B <_K D$ and $B \equiv_T \emptyset'$.

Consequences on minimal pairs

Observation:

If A is a c.e. real such that $\emptyset <_K A$ then there exists a c.e. set B with $\emptyset <_K B \leq_K A$.

Corollary

There are no minimal pairs of c.e. reals in the K-degrees.

Remark:

Barmpalias/Vlek showed that there exists a Δ^0_2 real X such that $\emptyset <_K X$ but there is no c.e. set B with $\emptyset <_K B \leq_K X$.

Broader consequences

The result provides an answer to:

How low can be the initial segment prefix-free complexity of a Turing complete computably enumerable set?

...also some information to:

Is there a pair of sequences X, Y which are not K-trivial but $\min\{K(X \upharpoonright_n), K(Y \upharpoonright_n)\} - K(n)$ has a constant upper bound?

Differences in measures of randomness

The structures of the Solovay degrees and the K-degrees of computably enumerable reals are not elementarily equivalent.

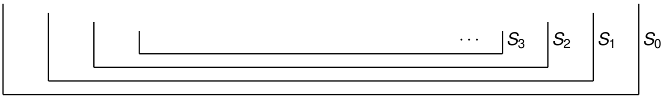
...same for the c.e. sets.

The structures of the C-degrees and the K-degrees of c.e. reals are not elementarily equivalent.

...same for the c.e. sets.

About the proof

- Dual to the decanter argument: K-trivials are incomplete;
- Same language: decanters, reusable descriptions, k-sets;
- Bounding the weight: $\mathtt{wgt}(M) \leq \sum_k k \cdot \mathtt{wgt}(S_k)$.



Infinite nested decanter model.

Using the same method. . .

If A is c.e. then $\{X \mid X \leq_K A\} \subseteq \Delta^0_2$.

How about uniformity?

Theorem (B. and A. Li)

The following are equivalent for a c.e. set A.

(a) *A is K-trivial;*

(b) *every set $X \leq_K A$ is truth-table reducible to $\emptyset'$;*

(c) *$\{X \mid X \leq_K A\}$ is uniformly computable in $\emptyset'$.*

Splitting and Mitoticity in $\leq_K$

Given a c.e. set can we split it into two c.e. sets of strictly less initial segment complexity?

Yes (B. and Sterkenburg) ... same for $\leq_{LK}$ (B., Lewis, Soskova)

Given a c.e. set can we split it into two c.e. sets of the same initial segment complexity?

- For $\leq_T$ a negative answer (Lachlan, Ladner)
- For $\leq_{LK}$ a negative answer (B. and Morphett)

Completely mitotic degrees: open for $\leq_{LK}$ and $\leq_K$.

Maximality in $\leq_K$

Is there a c.e. set whose initial segment complexity is maximal amongst the c.e. sets?

…same question for the global structure of $\leq_K$ (Miller and Yu)

Diluting sequences

Inserting 0s in an effective way reduces the complexity of a sequence.

This operation and its inversion was used by Ambos-Spies, Merkle, ... for nonmaximality in $\leq_{\mathrm{cl}}$.

Definition

An order is a strictly increasing computable function $f : \mathbb{N} \to \mathbb{N}$.

$X_f = \{f(n) \mid n \in X\}$.

X is K-invariant under f if $X \equiv_K X_f$.

K-resolute sequences

If X is K-trivial then $X_f \equiv_K X$ for all orders f.

Let f be an order. Every many-one degree contains a set X such that $X \equiv_K X_f$.

There exists a complete c.e. set A such that $A \equiv_K A_f$ for all orders f.

Every high [c.e.] degree contains a [c.e.] set A such that $A \equiv_K A_f$ for all orders f.

Open problem: Characterize their degrees.

Further topics

What is the algorithmic independence of c.e. sets?

Compare with the work of Levin, Calude and Zimmand on algorithmic independence.

How large is the class of sets with initial segment complexity bounded by the complexity of a c.e. set?

Compare with the work of Lutz and especially Hirschfeldt and Terwijin on Δ^0_2 measure.

References

- G. Barmpalias, Universal computably enumerable sets and initial segment prefix-free complexity.
- G. Barmpalias and A. Li, Kolmogorov complexity and computably enumerable sets.

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