Definability and weak reducibilities in algorithmic randomness

George Barmpalias

University of Amsterdam

Notre Dame, May 2010

Plan of the talk

- Classical vs weak reducibilities
- Exploiting the Σ_3^0-ness of a weak reducibility.
- When Σ_3^0 is not enough.
- Looking for definability/reducibility links in weak reducibilities.
- Exploiting local definability in weak reducibilities
- Minimal pairs of K degrees.

Classifying the continuum

- One way to classify reals according to a property is to define a reducibility and a degree structure.

- A reducibility is a pre-order that is used to compare the strength of the two reals according to the property in question.

  It partitions the continuum into equivalence classes, which are ordered according to the reducibility.

- Classical computability reducibilities: Turing, truth-table, many-one, etc.…

Reducibilities

- All classical reducibilities have an underlying map that reduces one real to another.

  $A \leq_R B$ iff there is a certain (perhaps partial) map $\Phi : 2^\omega \to 2^\omega$ such that $\Phi(B) = A$

- Such a map is usually an effective transformation that translates B into A.
- Most computability-theoretic reducibilities are Σ^0_3.

Measuring randomness and related properties

The study of relative randomness lead to the introduction of new reducibilities and degrees.

- Downey/Hirschfeldt/LaForte: Randomness and reducibility, J. Comput. System Sci. 68 (2004)
- Nies: Lowness properties and randomness, Adv. Math. 197 (2005)

Given a real. . .

How much random is it?

How much can it compress strings when it is used as an oracle?

How much random are the sequences that are random relative to it?

Associated reducibilities

$A \leq_K B$ if $K(A \restriction_n) \leq^+ K(B \restriction_n)$ for all n

$A \leq_C B$ if $C(A \restriction_n) \leq^+ C(B \restriction_n)$ for all n

$A \leq_{LK} B$ if $K^B(\sigma) \leq^+ K^A(\sigma)$ for all strings σ

$A \leq_{LR} B$ if every B-random is A-random

Miller showed that $\leq_{LR} = \leq_{LK}$.

Common features and differences with classic reducibilities

- Weak reducibilities do not have an underlying map which provides the reduction.

  That's why they are called **weak!**

  A fixed reduction may reduce many (even uncountably many) reals to a single real.

- But they are still Σ^0_3 (so each reduction has an index).

Emulating the classical theory

The fact that they are Σ^0_3 allows the use of (modified) classical methods from degree theory for their study.

Such an approach requires the translation of degree-theoretic parameters into a new context.

Not always straightforward (or even possible) but often successful.

Example of translations from classical theory

C.e. splitting in a degree structure: Every c.e. set of nonzero degree can be split into two disjoint c.e. sets whose degrees are incomparable and strictly less than the degree of the given set.

Theorem (Barmpalias/Lewis/Soskova 2006)

C.e. splitting holds in the LK degrees.

Theorem (Barmpalias/Sterkenburg 2010)

C.e. splitting holds in the K and the C degrees.

More examples

Randomness, Lowness and Degrees (B./Lewis/ Soskova, JSL 2008),

Π^0_1 classes, LR degrees and Turing degrees (B./Lewis/ Stephan, APAL 2008),

Weak reducibilities vs Turing reducibility

Turing degrees inside weak degrees.

Interactions are rather deep.

Many of the well known questions in the area are really about how Turing reducibility interacts with weak reducibilities.

...ML-coverability, ML-cuppability, etc.

Uncountable lower cones

- For some reals, the class of reals that are reducible to them is uncountable.

  Such reals are witnesses of the fact that in a weak reducibility the reduction is not always given by a map.

  Uncountably many reals may be reducible to a single real via a single index (reduction).

- In particular, if A is reducible to B, it is not always the case that A is definable in B (in any reasonable sense).

Uncountable LK lower cones

Question: *Given a real B, how many reals compress less efficiently than B?*

Uncountable *LK* lower cones

Question: *Given a real B, how many reals compress less efficiently than B?*

Formal question: *Given a real B, what is the cardinality of* $\{A \mid A \leq_{LK} B\}$*?*

Theorem

Countable if B is K-trivial (Nies 2003).

Uncountable if B is complete (Barmpalias/Lewis/Soskova 2006)

Uncountable if B is non-GL_2 (Barmpalias/Lewis/Soskova 2006)

Countable if B is low for Ω (Miller 2007)

Uncountable if B is Δ^0_2 and not K-trivial (Barmpalias 2008)

Definability and Turing reducibility

A set is computable iff it is Δ^0_1.

$A \leq_T B$ iff A is Δ^0_1 in B.

Post's theorem extends this link to arbitrary levels of the arithmetical complexity.

For example, A is Δ^0_2 in B iff $A \leq_T B'$.

If $A \leq_T B$ and B is Σ^0_1 then A is Δ^0_2.

Failure of definability in weak reducibilities

Turing reducibility is the largest which gives such definability implications.

This link between definability and reducibility breaks down in weak reducibilities.

This does not affect only the global degree structures based on weak reducibilities.

It also have striking implications in their countable substructures like the Σ_1^0 or the Δ_2^0 degrees.

Examples in the *LK* degrees

Theorem (Barmpalias (2009))

Every Δ^0_2 non-zero degree bounds a Σ^0_1 non-zero degree.

Theorem (Diamondstone (2010))

Any pair of low degrees has a low (c.e.) upper bound.

Looking for definability in weak reducibilities

Why?

Because then we can reproduce versions of arguments from classical computability to prove results about weak degree structures.

Examples in the *LK* degrees

How can we produce a minimal pairs in the LK degrees?

Theorem (Miller 2007)

If B is low for Ω and $A \leq_{LK} B$ then A is Δ^0_2 in B.

Applications in the *LK* structure

Corollary (Miller 2007)

There is a minimal pair.

Using additional results and a compactness argument...

Theorem (Barmpalias/Lewis/Ng 2009)

There is a minimal pair below the degree of $\emptyset'$.

A question on the LK degrees

- Every perfect set contains an uncountable antichain of Turing degrees (Sacks 1963)

- Every nonempty Π^0_1 set with no K-trivials contains an uncountable antichain of LK degrees (Barmpalias 2009).

Question: Does every perfect set of reals contain an uncountable antichain of LK degrees?

Another question on the LK degrees

Is there a minimal LK degree?

(It can't be Δ^0_2.)

Question (Downey and Hirschfedt 2006)

What is the optimal level of arithmetical complexity of minimal pair in the K-degrees?

History

There is a pair of Δ^0_4 sets that form a minimal pair in the K-degrees (Csima/Montalban 2006)

There is a pair of Σ^0_2 sets that form a minimal pair in the K-degrees (Merkle/Stephan 2007)

Definability in the K-degrees

Many lower cones are uncountable.

The definability/reducibility link survives in cones below Infinitely often K-trivial sets.

Definition

A set A is K-trivial on $M \subseteq \mathbb{N}$ if $K(A \upharpoonright_n) \leq^+ K(n)$ for each $n \in M$.

Infinitely often K-trivial sets

Proposition

- *Every* Σ^0_1 *set is i.o.* K*-trivial.*
- *Every (weakly) 1-generic is i.o.* K*-trivial.*
  I.o. K*-trivial sets form a comeager class.*
  There are uncountably many i.o. K*-trivials*
- *Every truth table degree contains an i.o.* K*-trivial.*
- *If a set does not compute a DNC function, it is i.o.* K*-trivial.*
- *There is a nonempty* Π^0_1 *class consisting of i.o. K-trivial sets that are not* K*-trivial.*

The coding theorem implies...

Proposition

If X is Δ^0_2 and i.o. K-trivial then $\{Y \mid Y \leq_K X\}$ is contained in Δ^0_2.

Proposition

The class of sets that have a Σ^0_1 bound in the K-degrees is contained in Δ^0_2.

Arithmetical complexity in the K-degrees

As in the Turing degrees...

Theorem (Barmpalias/Vlek 2010)

Let $n > 0$. There exists a nonzero Σ_n^0 degree that does not bound any nonzero Δ_n^0 degree.

Corollary

There exists a Σ_2^0 degree which forms a minimal pair with every nonzero Σ_1^0 degree.

A Δ^0_2 not bounding any Σ^0_1

Theorem (Barmpalias/Vlek 2010)

Let $\mathcal{F}$ be a Δ^0_2 family of sets/degrees. In the K-degrees there exists a Δ^0_2 nonzero degree that does not bound nonzero degrees in $\mathcal{F}$.

Corollary (Barmpalias/Vlek 2010)

In the K-degrees there exists a Δ^0_2 nonzero degree that does not bound any Σ^0_1 nonzero degree.

Comments

- Csima/Montalban used a gap function for K-triviality to produce a minimal pair of K-degrees.
- Such a function cannot be Δ^0_2 (B./Vlek 2010)

  Therefore their method cannot produce a Δ^0_2 minimal pair.
- Such gap functions are related to the so-called Solovay functions studied by Solovay, Bienvenu, Downey, Merkle. . . .

Question: Which K-degrees have countably many predecessors?

REFERENCE:

Barmpalias/Vlek, Kolmogorov complexity of initial segments of sequences and arithmetical definability. Preprint.

More at: `http://www.barmpalias.net`