

# Measures of relative complexity

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# Talking about problems

Many problems ask:

- ▶ for a YES or NO answer
- ▶ to determine a value  $y$



given an input value  $x$ .



**Data** and the solution to many problems can often be coded into finite or infinite **binary sequences**.

We work in the **Cantor space**.

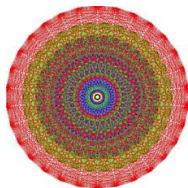


# Examples of problems

Presentations of **groups**

Diophantine **equations**

**Satisfiability** of propositional formulas



Verification of programs

Optimization

Raw data emanated from a physical source

# Reducibilities and hierarchies

Reducibilities are **preorders**  $\leq$ . They

- ▶ induce an equivalence relation denoting **identical complexity**
- ▶ **reorder the continuum** in terms of complexity
- ▶ are one of the main tools for **measuring complexity**



Hierarchies: **sequences of classes of increasing complexity**

Arithmetical hierarchy

Jump hierarchy

Polynomial hierarchy

Difference hierarchy

# Complexity of streams

An infinite binary sequence may be:

Theoretically computable

Definable (e.g. in arithmetic)

Feasibly computable

Incomputable (but possibly predictable)

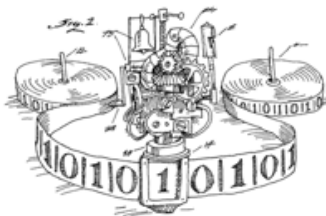
Random (with no algorithmic structure)



# Quantifying intractability

Some problems are intractable.

- ▶  $A$  is **computable in  $B$**
- ▶  $A$  is **definable in  $B$**
- ▶  $A$  is **feasibly computable in  $B$**
- ▶  $A$  is **random in  $B$**
- ▶  $A$  **less random than  $B$**



Complexity definitions **relativize to a parameter  $X$** .

A stream that is used as external information utilized in the course of a computation is called an **oracle**.

# Relative computability and degrees

$A \leq_T B$  denotes  $A$  is computable from  $B$

$A \equiv_T B$  denotes  $A \leq_T B$  and  $B \leq_T A$

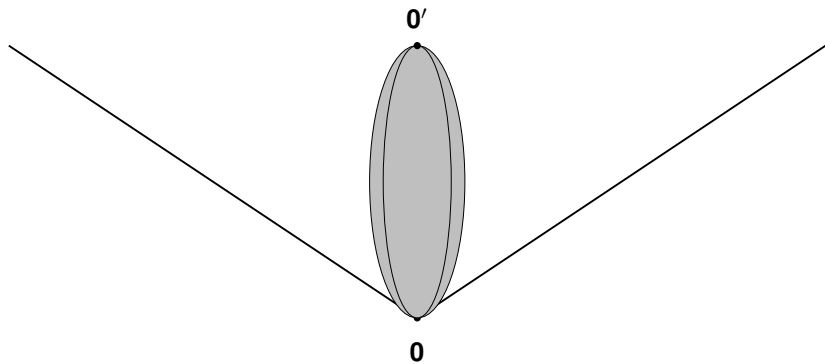
The equivalence classes are known as the **Turing degrees**.

They form a **partially ordered set**.

Two streams in the same degree are **equally tractable**.

A degree formalizes the concept of **information content**.

# Degrees of unsolvability



# Other classical reducibilities and degrees

- ▶ **Strong** reducibilities: truth-table, many-one, Lipschitz, . . .
- ▶ **Enumeration** reducibility
- ▶ **Arithmetical** reducibility



A Guiding Principle:

*Computability is some kind of definability and vice-versa.*

# People in degree theory



50s



60s,70s



70s, 80s



# Themes in degree theory

**Algebraic study:** embeddings, ideals, automorphisms

**Logical study:** decidability and complexity of the theory of degrees, definability.



*Global versus local structure and theory*

Computably enumerable sets and degrees

**Category and Measure** (genericity, randomness)



# Kolmogorov complexity of strings

Let  $|M|$  be the **size** of the machine  $M$ .

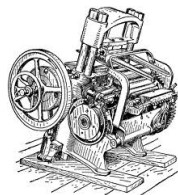
Let  $K_M(\sigma)$  be the complexity of  $\sigma$  w.r.t.  $M$ .

*The **complexity of  $\sigma$**  is the least sum  $|M| + K_M(\sigma)$  where  $M$  ranges over all machines.*

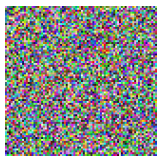
Let  $K(\sigma)$  denote the complexity of  $\sigma$ .



A string is **c-compressible** if it has a description that is shorter than its length by at least  $c$  bits.



# Algorithmic randomness



A stream  $X$  is **random** if there is a constant  $c$  such that  $K(X \upharpoonright_n) \geq n - c$  for all  $n$ .

This notion of randomness is **robust**:

- ▶ **Coincides with other approaches** (betting strategies, statistics)
- ▶ Random reals form a set of **measure 1**
- ▶ Meets **laws of large numbers**, normality etc.
- ▶ **Relativizes** giving randomness of various strengths

## On the other end: **trivial** initial segment complexity

The lowest possible initial segment complexity:  $K(0^n) \approx K(n)$ .



An interesting fact:

There are **noncomputable streams** with **trivial initial segment complexity**.

These streams are as simple as 000000... but **incomputable**!

# Measures of relative randomness

Segment-by-segment complexity comparison:

*$Y$  is at least as random as  $X$  if its initial segments have higher complexity than those of  $X$ .*

... if  $\forall n K(X \upharpoonright_n) < K(Y \upharpoonright_n) + c$  for some  $c$ .

Notation:  $X \leq_K Y$



- ▶  $\leq_K$  is a **weak reducibility**
- ▶ Induced degree structure: **the  $K$ -degrees**

# Differences and similarities of $\leq_K$ with $\leq_T$

- ▶ Same complexity as Turing reducibility
- ▶ Countable and uncountable degrees
- ▶ Lack of underlying reduction. . .  
(no algorithmic procedure connecting the two arguments)
- ▶ Countable and uncountable lower cones
- ▶ Structural differences
- ▶ Lack of a join operator. . . even upper bounds!

# Measuring compressing power

A string may be *X*-incompressible but *Y*-compressible.

*Y* can compress at least as well as *X* if it compresses every string more than *X* modulo a fixed number of bits.

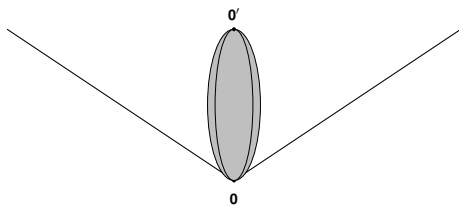
... if  $\exists c \forall \sigma K^Y(\sigma) \leq K^X(\sigma) + c$

... denoted by  $X \leq_{LK} Y$ .



$\leq_T$  implies  $\leq_{LK}$  (but not vice-versa).

# Degrees of compressibility: $LK$



- ▶ Same complexity as  $\leq_T$  (3rd level of arithmetical complexity)
- ▶ Countable degrees
- ▶ Countable and uncountable lower cones
- ▶ Lack of a natural supremum operator
- ▶ Structural differences with  $\leq_T$
- ▶ Large chains: a perfect set forming a  $\leq_{LK}$ -chain!

# Surprising coincidence: $\mathbf{0}_{LK} = \mathbf{0}_K$

*zero compressing power = trivial initial segment complexity*

... one of the main results in the last ten years in this area.



A stream *cannot compress* strings more than a computable oracle *if and only if* its *initial segments can be described as easily* as 0000...

# Reducibilities $\leq_K$ and $\leq_{LK}$ on random strings

For  $X, Y$  random:  $X \leq_K Y \iff Y \leq_{LK} X$

Informally, in the world of random streams...

*One stream is **more random** than another if and only if it has **less compressing power**.*



In particular...

*The **more random** a stream is, the **less oracle** power it has.*

... coded **information introduces structure** in a binary stream.

# Partial relativization and reducibilities

Some reducibilities come from **partial relativization of a lowness notion**.

*Example:  $X$  is low for  $K$  if it has zero compressing power.*



**Faithful relativization:**

' $X$  is low for random relative to  $Y$ ':  $X \oplus Y \leq_{LK} Y$  ... not transitive!

**Partial relativization:**  $X \leq_{LK} Y$  ... useful!

**Further examples:** Jump traceability, different randomness notions, ...

# Current research

Reducibilities associated with variations of randomness:

- ▶ Schnorr randomness
- ▶ Higher randomness: Borel and beyond

Measuring randomness of effective reals: **Solovay reducibility**

(For effective reals)

*Hardness of approximation equals degree of randomness*

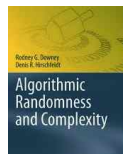
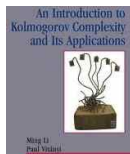
Various basic questions remain open.

*Is there a maximal  $K$ -degree?*

Research in local substructures is also an interesting direction.

**Computably enumerable sets:** Density? Upper bounds ?

# Further reading



- ▶ Li-Vitanyi, **An introduction to Kolmogorov Complexity and its applications**, Springer-Verlag.
- ▶ Nies, **Computability and Randomness**, Oxford Press.
- ▶ Downey and Hirschfeldt, **Algorithmic randomness and complexity**, Springer-Verlag.

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