Randomness, Lowness and Degrees

A study of the *LR* degrees

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Overview

- There are a number of lowness notions for oracles in computability theory and algorithmic randomness.
- For example:
  - low, superlow oracles with respect to the jump
  - oracles of hyperimmune-free Turing degree (**0**-dominated oracles)
  - low for random oracles
  - low for Ω (the halting probability) oracles
  - Weakly low for K oracles
- These are properties asserting that an oracle is weak in a certain sense.
- In this work we study lowness in the context of a relevant reducibility called $\leq_{LR}$
- and determine various connections with a number of lowness notions and the Turing computation

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The Cantor space

- 2^ω is the space of infinite binary strings: the *reals*
- $2^{<\omega}$ is the space of finite binary strings
- The standard topology on 2^ω is induced by the basic open sets: $[\sigma] = \{\sigma X \ : \ X \in 2^\omega\}$ for all $\sigma \in 2^{<\omega}$.
- Lebesgue measure on the Cantor space: the measure of a basic open set $[\sigma]$ is $\mu([\sigma]) = 2^{-|\sigma|}$

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Martin-Löf Randomness

- Identify finite binary strings with intervals in $[0, 1]$: $\sigma \rightarrow [\sigma]$
- Prefix-free sets of finite binary strings correspond to independent (basic open) sets of reals

Definition

A Martin-Löf test $\mathcal{M}$ is a uniform sequence (E_i) of c.e. sets of binary strings such that $\mu(E_i) \leq 2^{-i}$. A real α avoids $\mathcal{M}$ if some for i, $\alpha \notin E_i$. A real number is called random if it avoids all Martin-Löf tests. W.l.o.g. assume $E_{i+1} \subset E_i$.

- Martin-Löf tests and randomness relativize to any oracle.
- we say n-random for $\emptyset^n$-random

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A fact about random sets

If P is a Π^0_1 class and $\beta \in P$ say that I is an independent set for β in P if for every 0-1 combination of digits on positions $n \in I$ in β the resulting real also belongs to P.

Theorem (Joe Miller)

For every Π^0_1 class P and every random real $\beta \in P$ there is an independent set for β in P which is computable from β'.

Applications

Theorem

- *For every $m, n \geq 1$ there exist sets which are n-random and which are properly m-c.e. in $\emptyset^{(n)}$.*
- *In particular for every $n \geq 1$ there exists an n-random real in $\Sigma^0_{n+1} - \Delta^0_{n+1}$*
- *Thus the arithmetical hierarchy theorem holds for random sets*
- *Every PA degree (i.e. computing a complete extention of Peano Arithmetic) is the join of two random degrees.*

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Relative randomness and lowness

- *A* is *low for random* if every random is *A*-random.
- Relativizing we get: $A \leq_{LR} B$ if every B-random is A-random.
- $\leq_{LR}$ is transitive, Σ^0_3 and it contains $\leq_T$.
- Induced degrees: $A \equiv_{LR} B$ if the A-randoms coincide with the B-randoms

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Standard Conventions

- we write $\mu(U)$ for the measure of the corresponding class of reals
- all subset relations $U \subset V$ where U, V are sets of strings actually refer to the corresponding classes of reals
- Boolean operations on sets of strings actually refer to the same operations on the corresponding classes of reals.
- all sets of strings will be prefix free. Even when effective enumerations are concerned one can assume this without loss of effectiveness.

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Basic fact (Kjos-Hansen)

The following are equivalent:

- $A \leq_{LR} B$
- For every $\Sigma_1^{0,A}$ class T^A of measure < 1 there is a $\Sigma_1^{0,B}$ class V^B of measure < 1 such that

$$T^A \subseteq V^B.$$

- For some member U^A of a universal Martin-Löf test relative to A there is $V^B \in \Sigma_1^{0,B}$ with $\mu V^B < 1$ and

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Structure and Properties

- Each degree contains countably many elements (Nies).
- There is a least degree containing the low for random reals. Some of them are not computable (Kučera).
- It is not known if there is a least upper bound for any two degrees.
- The usual $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ is not a supremum (Nies).

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- transfering techniques from the c.e. Turing degrees to the study of c.e. *LR*-degrees
- arguments are usually more interesting than the Turing case
- interactions between cost function arguments with the classic techniques (Sacks restraints, coding)
- interesting examples of infinitary cost arguments

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A Splitting theorem for the c.e. LR degrees

Theorem

If A is c.e. and not low for random then there are c.e. B, C such that

- $B \cap C = \emptyset$
- $B \cup C = A$
- $B \not\leq_{LR} C$ and $C \not\leq_{LR} B$.

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Weak Density for the c.e. LR degrees

The fact that $\oplus$ is not a supremum operator creates problems in proving density. However a modification of the Sacks density argument still allows us to show the following:

Theorem

If B, C are r.e. and $B \leq_T C$, $C \not\leq_{LR} B$ then there is a r.e. A such that $B \leq_T A \leq_T C$, $C \not\leq_{LR} A$ and $A \not\leq_{LR} B$.

Note that this implies *downward and upward density* for the c.e. LR degrees.

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Structure of c.e. Turing degrees inside a c.e. *LR* degree

An example of Sacks coding and restraints in the *LR* degrees:

Theorem

Let **a** *be an incomplete noncomputable c.e. Turing degree. Then*

- *there is a c.e. Turing degree strictly below* **a** *which is in the same LR degree*
- *there is a c.e. Turing degree strictly above* **a** *which is in the same LR degree*
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More about $\leq_T$ inside $\leq_{LR}$

A further relation between $\leq_T$ and $\leq_{LR}$ is estabilished in the following.

- (B. and Montalbán) There is a c.e. A which is half of a minimal pair in the c.e. Turing degrees and $A \equiv_{LR} \emptyset'$
- (B. and Anthony Morphett) There is a c.e. A such that $A \oplus B <_T \emptyset'$ for all c.e. $B <_T \emptyset'$ and $A \equiv_{LR} \emptyset'$

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$2^{\aleph_0}$ predecessors

Definition
The set A is generalized non-low_2 ($\overline{GL_2}$) if $A'' >_T (A \oplus \emptyset')'$.

Theorem
The LR degree of the halting problem bounds $2^{\aleph_0}$ LR degrees. More generally, if α is $\overline{GL_2}$ then in the LR degrees the degree of α bounds $2^{\aleph_0}$ degrees.

By a cardinality argument we get the following:

Theorem
There are pairs of comparable LR degrees such that the Turing degrees they contain are pairwise incomparable. In fact, if A is $\overline{GL_2}$ then there is $B <_{LR} A$ such that the pair A, B has this property.

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More reals with $2^{\aleph_0}$ *LR*-predecessors

Theorem

- *Every real of PA-degree has uncountably many LR-predecessors*
- *There is a superlow c.e. set A such that* $\{B \mid B \leq_{LR} A\}$ *contains a perfect* Π^0_1 *class.*

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Countable predecessor property

- Who has it?
- Joe Miller suggested the idea that highly random reals resemble trivial reals (without being $\leq_{LR} \emptyset$).
- Weakly Low for K is a lowness notion which is more general than low for randomness.

Theorem (Joe Miller)

- *Every $\emptyset''$-random (2-random) is weakly low for K.*
- *If $A \leq_{LR} B$ and B is weakly low for K then $A \leq_T B'$.*
- *So almost all reals have only countably many predecessors.*

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Antichains of size $2^{\aleph_0}$

Theorem

There exists an antichain of cardinality $2^{\aleph_0}$ in the LR degrees.

Structure of Turing degrees inside an *LR* degree

Theorem

Let **a** *be a Turing degree.*

- *there is a Turing degree strictly above* **a** *which is in the same LR degree as* **a***.*
- *if* **a** $>$ **0** *there is a Turing degree incomparable with* **a** *which is in the same LR degree.*
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Low for Ω, *LR* degrees, and Π^0_1 classes

An oracle A is called *low for* Ω if Ω is A-random.

Theorem

There is a Π^0_1 class of GL_1 reals $\leq_{LR} \emptyset'$ which contains no low for random reals.

By the low for Ω basis theorem we get the following

Corollary

There is a low for Ω real which is LR-below Ω and it is not low for random.

This means that 'bases for randomness' with respect to $\leq_{LR}$ do not have to be low for random. This contrasts the situation in $\leq_T$.

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Let A be low for Ω. Then every perfect Π^0_1 class has a member B such that $B \not\leq_{LR} A$.

The proof involves a double use of van Lambalgen's theorem.
The following demonstrates how far is $\oplus$ from defining the supremum operation in the LR degrees.

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By the previous theorem and the hyperimmune-free basis theorem we have

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There is a hyperimmune-free real which is $\leq_{LR} \mathbf{0}'$ and not low for random.

However,

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For every set A there is a set B of hyperimmune-free degree such that $B \not\leq_{LR} A$.

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- Is there a c.e. non-computable set which is computable by all c.e. *LR*-complete sets ?
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