Computably Enumerable Splittings, Randomness and Lowness

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8th December 2007

Relative Randomness

Given two oracles A, B it is natural to ask how the class of random sequences relative to A relate with the class of random sequences relative to B.

- This relation is formally represented by the LR reducibility:
- We say that A is LR reducible to B if every sequence which is not random relative to A, is not random relative to B.
- We say that $A \equiv_{LR} B$ if the classes of A-randoms and B-randoms coincide.

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Lowness

- There are a number of lowness notions for oracles in computability theory and algorithmic randomness.
- For example:
  - low, superlow oracles with respect to the jump
  - oracles of hyperimmune-free Turing degree ($\mathbf{0}$-dominated oracles)
  - low for random oracles
  - low for Ω (the halting probability) oracles
  - Weakly low for K oracles
- These are properties asserting that an oracle is weak in a certain sense.
- We have studied lowness in the context of the LR reducibility
- and have determined various connections with a number of lowness notions and the Turing computation

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The Cantor space

- 2^ω is the space of infinite binary strings: the *reals*
- $2^{<\omega}$ is the space of finite binary strings
- The standard topology on 2^ω is induced by the basic open sets: $[\sigma] = \{\sigma X \ : \ X \in 2^\omega\}$ for all $\sigma \in 2^{<\omega}$.
- Lebesgue measure on the Cantor space: the measure of a basic open set $[\sigma]$ is $\mu([\sigma]) = 2^{-|\sigma|}$

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Martin-Löf Randomness

- Identify finite binary strings with intervals in $[0,1]$: $\sigma \to [\sigma]$
- Prefix-free sets of finite binary strings correspond to independent (basic open) sets of reals

Definition

A Martin-Löf test $\mathcal{M}$ is a uniform sequence (E_i) of c.e. sets of binary strings such that $\mu(E_i) \leq 2^{-i}$. A real α avoids $\mathcal{M}$ if some for i, $\alpha \notin E_i$. A real number is called random if it avoids all Martin-Löf tests. W.l.o.g. assume $E_{i+1} \subset E_i$.

- Martin-Löf tests and randomness relativize to any oracle.
- we say n-random for $\emptyset^{n-1}$-random

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Relative randomness and lowness

- A is *low for random* if every random is A-random.
- Relativizing we get: $A \leq_{LR} B$ if every B-random is A-random.
- $\leq_{LR}$ is transitive, Σ^0_3 and it contains $\leq_T$.
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Standard Conventions

- we write $\mu(U)$ for the measure of the corresponding class of reals
- all subset relations $U \subset V$ where U, V are sets of strings actually refer to the corresponding classes of reals
- Boolean operations on sets of strings actually refer to the same operations on the corresponding classes of reals.
- all sets of strings will be prefix free. Even when effective enumerations are concerned one can assume this without loss of effectiveness.

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Basic fact (Kjos-Hansen)

The following are equivalent:

- $A \leq_{LR} B$
- For every $\Sigma_1^{0,A}$ class T^A of measure < 1 there is a $\Sigma_1^{0,B}$ class V^B of measure < 1 such that

$$T^A \subseteq V^B.$$

- For some member U^A of a universal Martin-Löf test relative to A there is $V^B \in \Sigma_1^{0,B}$ with $\mu V^B < 1$ and

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Basic Structure and Properties

- Each degree contains countably many elements (Nies).
- There is a least degree containing the low for random reals. Some of them are not computable (Kučera).
- It is not known if there is a least upper bound for any two degrees.
- The usual $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ is not a supremum (Nies).
- The measure of upper cones of LR degrees is 0 (an application of van Lambalgen's theorem by Frank Stephan)

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Current Knowledge on the *LR* degrees

- Structure of Turing degrees inside an *LR* degree, in the c.e. case and globally.
- Uncountable and countable lower *LR*-cones.
- cone avoidance, weak density for the c.e. *LR* degrees, c.e. splittings
- uncountable antichains of *LR* degrees, priority methods and forcing arguments.

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A Splitting theorem for the c.e. LR degrees

Given a c.e. oracle A and a c.e. splitting of it, it is natural to ask what notions of randomness the components of it induce, and how these are related to each other and to A-randomness.

By known facts, if a c.e. A is low for random, then any splitting of it will produce oracles inducing the same notion of randomness.

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The following theorem says that if A is c.e. and not low for random, then there is a nontrivial c.e. splitting of it.

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If A is c.e. and not low for random then there are c.e. B, C such that

- $B \cap C = \emptyset$
- $B \cup C = A$
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- Say that A is of almost everywhere dominating degree if it computes a function which dominates almost all other functions.
- This is a "highness" notion.
- It is well known that given a c.e. set, it is of almost everywhere dominating degree iff $A \equiv_{LR} \emptyset'$.
- Then the above theorem says that there is a c.e. set of almost everywhere dominating degree which cannot be split into two c.e. parts of almost everywhere dominating degree.

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About the proof

- a finite injury argument, with some coding for $A \equiv_T \emptyset'$.
- Coding may be avoided by a standard "reverse reasoning" about the complexity of the set.
- (e.g. Friedberg-Muchnik automatically produces a pair of Turing degrees joining to $\emptyset'$).
- The idea is to construct A in such a way that any splitting of it B, C which tries to achieve $U^A \subset V^B$ and $U^A \subset V^C$ fails on at least one of the clauses.
- We try to "blow-up" V^B, V^C with a lot of measure, so that at least one of them cannot keep up with the restriction of having measure < 1.
- This argument falls into a group of arguments in the LR degrees where ideas and techniques from the Turing degrees can be transferred to the LR case.
- This is not always straightforward, and sometimes even not possible.

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Theorem

A c.e. set A is low for random iff it computes a c.e. set which cannot be split into two c.e. sets of the same LR degree.

- One direction is straightforward.
- The other direction is a genuine *LR*-permitting argument, the first known one.
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- perform the previous argument below A.
- As we try to blow-up V^B, V^C with measure, the condition that $A \not\leq_{LR} \emptyset$ guaranties that enough measure will be permitted by A, so that we succeed in over-blowing one of them.
- note that if $A \not\leq_{LR} \emptyset$ then any attempt to cover U^A with a Σ^0_1 class V of measure < 1 fails, i.e. if $U^A \subseteq V$ then $\mu(V) = 1$.
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