Computably Enumerable Splittings, Randomness and Lowness

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Relative Randomness

Given two oracles $A$, $B$ it is natural to ask how the class of random sequences relative to $A$ relate with the class of random sequences relative to $B$.

- This relation is formally represented by the $LR$ reducibility:
- We say that $A$ is $LR$ reducible to $B$ if every sequence which is not random relative to $A$, is not random relative to $B$.
- We say that $A \equiv_{LR} B$ if the classes of $A$-randoms and $B$-randoms coincide.
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Lowness

• There are a number of lowness notions for oracles in computability theory and algorithmic randomness.

• For example:
  - low, superlow oracles with respect to the jump
  - oracles of hyperimmune-free Turing degree (\(0\)-dominated oracles)
  - low for random oracles
  - low for \(\Omega\) (the halting probability) oracles
  - Weakly low for \(K\) oracles

• These are properties asserting that an oracle is weak in a certain sense.

• We have studied lowness in the context of the \(LR\) reducibility

• and have determined various connections with a number of lowness notions and the Turing computation
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The Cantor space

- $2^\omega$ is the space of infinite binary strings: the *reals*
- $2^{<\omega}$ is the space of finite binary strings
- The standard topology on $2^\omega$ is induced by the basic open sets: $[\sigma] = \{\sigma X : X \in 2^\omega\}$ for all $\sigma \in 2^{<\omega}$.
- Lebesgue measure on the Cantor space: the measure of a basic open set $[\sigma]$ is $\mu([\sigma]) = 2^{-|\sigma|}$
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Martin-Löf Randomness

- Identify finite binary strings with intervals in $[0, 1]$: $\sigma \rightarrow [\sigma]$
- Prefix-free sets of finite binary strings correspond to independent (basic open) sets of reals

Definition
A Martin-Löf test $M$ is a uniform sequence $(E_i)$ of c.e. sets of binary strings such that $\mu(E_i) \leq 2^{-i}$. A real $\alpha$ avoids $M$ if some for $i$, $\alpha \notin E_i$. A real number is called random if it avoids all Martin-Löf tests. W.l.o.g. assume $E_{i+1} \subset E_i$.

- Martin-Löf tests and randomness relativize to any oracle.
- we say $n$-random for $\emptyset^{n-1}$-random
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Relative randomness and lowness

- $A$ is *low for random* if every random is $A$-random.
- Relativizing we get: $A \leq_{LR} B$ if every $B$-random is $A$-random.
- $\leq_{LR}$ is transitive, $\Sigma^0_3$ and it contains $\leq_T$.
- Induced degrees: $A \equiv_{LR} B$ if the $A$-randoms coincide with the $B$-randoms.
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Standard Conventions

• we write $\mu(U)$ for the measure of the corresponding class of reals
• all subset relations $U \subset V$ where $U, V$ are sets of strings actually refer to the corresponding classes of reals
• Boolean operations on sets of strings actually refer to the same operations on the corresponding classes of reals.
• all sets of strings will be prefix free. Even when effective enumerations are concerned one can assume this without loss of effectiveness.
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Basic fact (Kjos-Hansen)

The following are equivalent:

- $A \leq_{LR} B$
  - For every $\Sigma^0_1,A$ class $T^A$ of measure $< 1$ there is a $\Sigma^0_1,B$ class $V^B$ of measure $< 1$ such that
    $$T^A \subseteq V^B.$$  
  - For some member $U^A$ of a universal Martin-Löf test relative to $A$ there is $V^B \in \Sigma^0_1,B$ with $\mu V^B < 1$ and
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Basic Structure and Properties

- Each degree contains countably many elements (Nies).
- There is a least degree containing the low for random reals. Some of them are not computable (Kučera).
- It is not known if there is a least upper bound for any two degrees.
- The usual $A \oplus B = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in B\}$ is not a supremum (Nies).
- The measure of upper cones of $LR$ degrees is 0 (an application of van Lambalgen’s theorem by Frank Stephan).
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Current Knowledge on the $LR$ degrees

- Structure of Turing degrees inside an $LR$ degree, in the c.e. case and globally.
- Uncountable and countable lower $LR$-cones.
- Cone avoidance, weak density for the c.e. $LR$ degrees, c.e. splittings
- Uncountable antichains of $LR$ degrees, priority methods and forcing arguments.
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A Splitting theorem for the c.e. LR degrees

Given a c.e. oracle $A$ and a c.e. splitting of it, it is natural to ask what notions of randomness the components of it induce, and how these are related to each other and to $A$-randomness.

By known facts, if a c.e. $A$ is low for random, then any splitting of it will produce oracles inducing the same notion of randomness.
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By known facts, if a c.e. $A$ is low for random, then any splitting of it will produce oracles inducing the same notion of randomness.
The following theorem says that if $A$ is c.e. and not low for random, then there is a nontrivial c.e. splitting of it.

Theorem
If $A$ is c.e. and not low for random then there are c.e. $B, C$ such that

- $B \cap C = \emptyset$
- $B \cup C = A$
- $B \not\leq_{LR} C$ and $C \not\leq_{LR} B$. 
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Splitting inside an $LR$ degree

Given a c.e. set, can we split it into two c.e. parts which induce the same notion of relative randomness? Not always.
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Introduction

**Splitting inside an LR degree**

**Theorem**

_There is a c.e. set A such that for all c.e. splittings B, C of it, A \not\leq_{LR} B or A \not\leq_{LR} C. Moreover A can be chosen such that A \equiv_T \emptyset'._

- Say that A is of almost everywhere dominating degree if it computes a function which dominates almost all other functions.
- This is a “highness” notion.
- It is well known that given a c.e. set, it is of almost everywhere dominating degree iff A \equiv_{LR} \emptyset'.
- Then the above theorem says that there is a c.e. set of almost everywhere dominating degree which cannot be split into two c.e. parts of almost everywhere dominating degree.
Splitting inside an $LR$ degree

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About the proof

- a finite injury argument, with some coding for $A \equiv_T \emptyset'$.  
- Coding may be avoided by a standard “reverse reasoning” about the complexity of the set.  
- (e.g. Friedberg-Muchnik automatically produces a pair of Turing degrees joining to $\emptyset'$).  
- The idea is to construct $A$ in such a way that any splitting of it $B, C$ which tries to achieve $U^A \subset V^B$ and $U^A \subset V^C$ fails on at least one of the clauses.  
- We try to “blow-up” $V^B, V^C$ with a lot of measure, so that at least one of them cannot keep up with the restriction of having measure $< 1$.  
- This argument falls into a group of arguments in the LR degrees where ideas and techniques from the Turing degrees can be transferred to the LR case.  
- This is not always straightforward, and sometimes even not possible.
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Theorem
A c.e. set $A$ is low for random iff it computes a c.e. set which cannot be split into two c.e. sets of the same LR degree.

- One direction is straightforward.
- The other direction is a genuine $LR$-permitting argument, the first known one.
- Non-low for random sets “permit measure”.
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- perform the previous argument below $A$.
- As we try to blow-up $V^B$, $V^C$ with measure, the condition that $A \not\subseteq_{LR} \emptyset$ guaranties that enough measure will be permitted by $A$, so that we succeed in over-blowing one of them.
- note that if $A \not\subseteq_{LR} \emptyset$ then any attempt to cover $U^A$ with a $\Sigma^0_1$ class $V$ of measure $< 1$ fails, i.e. if $U^A \subseteq V$ then $\mu(V) = 1$.
- This means that $A$ has a way to blow large amounts of measure into sets.
- Then we need to map the measure in $V$ onto the measure we actually want to put into $V^B$, $V^C$, and use the fact that $\mu(V) = 1$ in order to argue that at least one of $V^B$, $V^C$ will be large.
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- The above argument does not combine with coding.
- Does every non-trivial LR degree contain a c.e. set which is not splittable in the same LR degree?
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