On the number of infinite sequences with trivial initial segment complexity

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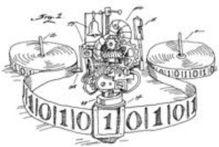
Kolmogorov complexity of strings

Kolmogorov, 1960s:

“The complexity of a binary string is the length of its shortest description.”

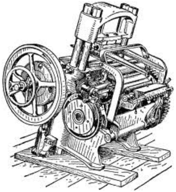
Descriptions given in an algorithmic way:

If M is a Turing machine and $M(\sigma) = \tau$ then σ is an M-description of τ.



Kolmogorov complexity of strings: Machines

Let $|M|$ be the **size** of the machine M.



Let $C_M(\sigma)$ be the complexity of σ w.r.t. M.

The complexity of σ is the least sum $|M| + C_M(\sigma)$ where M ranges over all machines.

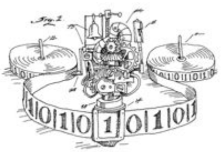
Let $C(\sigma)$ denote the complexity of σ.

By universality, Kolmogorov complexity is an effective concept.

Kolmogorov complexity of strings: Optimality

There is a machine that gives optimal descriptions:

On a given input, interpret the first few bits as a description of a machine. . .



. . . and simulate that machine on the tail of the original input.

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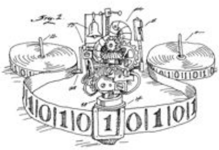


In other words... $U(0^e 1\sigma) = M_e(\sigma)$ *where M_e is the eth machine.*

Let $C(\sigma)$ denote the Kolmogorov complexity of σ.

Examples:

$C(\sigma) \leq |\sigma|$ and $C(n) \leq \log n$

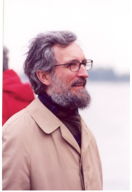
Randomness of infinite sequences

A string is *c*-incompressible if it has no description that is shorter than its length by at least *c* bits.

Kolmogorov sought to define:

“A binary stream is random if for some constant *c* all of its initial segments are *c*-incompressible.”

Unfortunately, Martin-Löf showed that

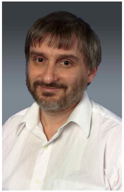


For every constant *c* and every stream *X* has a *c*-compressible initial segment.

Chaitin and Levin, 1970s

They observed that

Given a string, one can recover information from the bits of the string but also from its length.



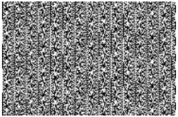
More faithful complexity measure via prefix-free machines:

Prefix-free machines cannot extract information from the length of a string.

Let $K(\sigma)$ be the prefix-free complexity of σ.

$K(\sigma) \leq |\sigma| + K(|\sigma|)$ and $K(n) \leq 2\log n$.

Algorithmic randomness



A stream X is random if there is a constant c such that $K(X \restriction_n) \geq n - c$ for all n.

This notion of randomness is robust:

- Coincides with other approaches (betting strategies, statistics)
- Random reals form a set of measure 1
- Meets laws of large numbers, normality etc.

Trivial sequences

A stream is random if it has high initial segment complexity.

> *To describe the first n bits of the sequence you need to use n bits (modulo a constant)*

On the other end of the spectrum:

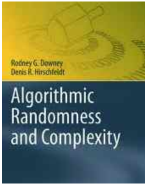
> *A stream is trivial if the complexity of its first n bits is as low as the complexity of* 0^n.

Solovay 1975

Chaitin asked if there are non-computable streams whose initial segment complexity is as low as a computable stream.

Solovay gave a positive answer.

Draft of a paper (or series of papers) on Chaitin's work. Unpublished notes, May 1975. 215 pages.



Solovay's work entirely included...

...in Downey/Hirschfeldt book

The world of K-trivial streams

$\rightsquigarrow$ Computable from the halting problem i.e. Δ^0_2 (Chaitin 70s)

$\rightsquigarrow$ Incomplete, and in fact low (Downey/Hirschfeldt/Nies/Stephan)

$\rightsquigarrow$ Provide a 'natural' solution to Post's problem.

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$$A = \{n \mid \exists e, s\, (\underbrace{W_{e,s} \cap A_s = \emptyset \ \wedge\ n > 2e \ \wedge n \in W_{e,s}}_{\text{Post's simple set}} \wedge \textstyle\sum_{n<j<s} 2^{-K_s(j)} < 2^{-e})\}$$

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$\rightsquigarrow$ Downward closed under oracle computation (Nies 2005)

$\rightsquigarrow$ Form an ideal in the Turing degrees.

Cumulative hierarchy of K-trivial streams

A stream X is K-trivial if $K(X \upharpoonright_n) \leq K(n) + c$ for all n, some c.

K-trivial streams are stratified in a hierarchy of length ω

. . . whose c-level contains the K-trivial streams with constant c.

Question by Downey, Miller, Nies, Yu

What is the arithmetical complexity of the map G which takes c to the number of K-trivial strings with constant c?

. . . or equivalently

How many quantifiers are needed in order to define G in arithmetic?

. . . or equivalently

How hard is to compute G?

Basic facts about G, by DMNY

- Computable from $\mathbf{0}^{(3)} \ldots$ i.e. Δ^0_4
- Not computable i.e. not Δ^0_1
- Not computable from the halting problem, i.e. not Δ^0_2

Is it computable from $\mathbf{0}^{(2)}$ i.e. is it Δ^0_3?

The classes of K_c-trivial streams

- They are uniformly Π^0_1 in the halting set
- The set of infinite paths through a $\mathbf{0}'$-computable tree.
- The width of these trees is computably bounded since

$$|\{\sigma \in 2^n \mid K(\sigma) \leq K(|\sigma|) + c\}| < 2^c$$

. . . by the coding theorem

Number of paths through trees of bounded width

- The number of infinite paths through a tree T with bounded width can be computed from T''.
- This is optimal!
- If a family of trees is computable from a low$_2$ oracle A then the number of paths is computable from $\mathbf{0}^{(2)}$.
- Oracle A is low$_2$ if A'' is computable from $\mathbf{0}^{(2)}$; $\Sigma^0_2(A) \subseteq \Delta^0_3$.

Representing the K_c-trivial classes with simpler trees

Theorem (B. and Tom Sterkenburg)

Given a Δ^0_2 tree T which only has K_c-trivial paths we can compute the index of another Δ^0_2 tree which is K-trivial and has the same infinite paths as the original tree.

The new trees have trivial initial segment complexity.

Fact: $\mathbf{0}^{(2)}$ can compute a low$_2$ index of a K_c-trivial stream given c and the Δ^0_2 index of the stream.

Computation of $G(c)$ from $\mathbf{0}^{(2)}$

- Get the index of the original Δ^0_2 tree representing the class K_c-trivial.
- Compute the index of the K-trivial tree representing this class.
- Use $\mathbf{0}^{(2)}$ to compute a low$_2$-ness index of the new tree.
- Use $\mathbf{0}^{(2)}$ again to compute the number of infinite paths through this tree.
- This is $G(c)$

A related class: low for K streams

If a computer is given access to a powerful oracle, it will achieve better compression for many strings.

X is called low for K if the prefix-free complexity relative to X is the same as the unrelativized prefix-free complexity.

> ......*if X cannot compress strings more than a constant number of bits.*

> ......*if as far as prefix-free complexity is concerned, it is not better than a computable oracle.*

This class was defined by Muchnik in 1999, who also exhibited non-computable elements in it.

Hierarchy of low for K and complexity

- Low for K streams are stratified in a cumulative hierarchy of finite classes.
- Hirschfeldt and Nies showed that they coincide with the K-trivial streams.
- Our methodology applies to this class, showing that

... the corresponding function giving the cardinality of the hierarchy classes is Δ^0_3.

Applications

A consequence of the main result is that $\mathbf{0}''$ can obtain the indices of the K_c-trivial strings.

This can be used to show that a number of K-related objects have lower complexity.

For example, gap functions for K-triviality.

Gap functions for K-triviality

These are non-decreasing unbounded functions f such that

$$\forall n\,[K(X \upharpoonright_n) \leq K(n) + f(n) + c] \Rightarrow X \text{ is } K\text{-trivial.}$$

- Constructed by Csima and Montalbán in 2006
- Used to obtain minimal pairs in the degrees of randomness
- Complexity: Δ^0_4
- Downey raised the question about their complexity

Complexity of gap functions

Theorem (B. and Martijn Baartse)

If f is Δ^0_2 unbounded and non-decreasing then there are uncountably many streams X such that

$$K(X \upharpoonright_n) \leq K(n) + f(n) \quad \textit{for all } n.$$

Theorem (B. and Martijn Baartse)

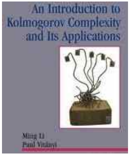
There is a Δ^0_3 gap function for K-triviality.

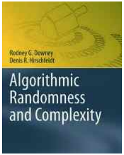
References

- Barmpalias/Sterkenburg, On the number of infinite sequences with trivial initial segment complexity
- Barmpalias/Baartse, On the gap between trivial and nontrivial initial segment prefix-free complexity

- Webpage: `http://www.barmpalias.net`

General references





- Li-Vitanyi, An introduction to Kolmogorov Complexity and its applications, Springer-Verlag.
- Nies, Computability and Randomness, Oxford Press.
- Downey and Hirschfeldt, Algorithmic randomness and complexity, Springer-Verlag.