Degrees of unsolvability and degrees of compressibility

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Plan of the talk

- Introduction
- Triviality
- Relativization
- Structure of LK degrees
- Main result and applications
- Further questions
- References

Computability and Randomness

- The theory of computable sets and numbers (Turing 1936) naturally led to the theory of relative computation and unsolvability (Turing 1939, Post 1943)

In the same way...

- the study of the 'descriptive' complexity of strings and streams has naturally lead to the study of relativized complexity

Oracle machines

- Relative computability aims at providing measures to compare and study objects according to their information content
- In the same way, relative randomness aims at comparing mathematical objects with respect to the randomness-related properties they might have.
- In this transition from the effective to the relativized theory, Turing machines get equipped with a source of external information.

Triviality

- The notion of a computable set is central in computability theory.
- Recent work on effective randomness suggests that the notion of K-triviality is of analogous importance in this area.

A is K-trivial if its initial segments have trivial complexity: $K(A \upharpoonright n) \leq^+ K(0^n)$, for all $n \in \mathbb{N}$.

More triviality

Let K^X be the prefix-free complexity relative to oracle X

- K^X is based on a machine which can use external information X for compressing.
- Therefore it may compress more effectively than a machine without an oracle.

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A is low for K if it does not have useful information for compressing strings: $K^A(\sigma) \leq^+ K(\sigma)$, for all strings σ.

- Nies/Hirschfeldt showed that K**-trivials** $=$ **low for** K
- Downey/Hirschfeldt/Laforte introduced the following measure: $A \leq_K B$ if $K(A \upharpoonright n) \leq^+ K(B \upharpoonright n)$ for all n.
- Similarly, Nies defined: $A \leq_{LK} B$ if $K^B(\sigma) \leq^+ K^A(\sigma)$ for all strings σ.
- Then $A \equiv_{LK} B$ means that A, B have the same power for compression

Compare: $A \equiv_T B$ means that A, B have the same power for computation, same information.

Facts

- Miller showed that $A \equiv_{LK} B$ iff A-randoms $=$ B-randoms
- The LK measure is a natural extension of the relative Turing measure.

Remarkable fact: There is a non-computable A such that $A \equiv_{LK} \emptyset$.

Structure theory

- The relation $\leq_{LK}$ and its connections with $\leq_T$ have been investigated in the last 5 years.
- A lot of the popular open problems in randomness today are about the relationship between $\leq_{LK}$, $\leq_T$.
- $\leq_{LK}$ has uncountable lower cones!
- However locally, the algebraic structure of $\leq_{LK}$ and $\leq_T$ looked the same.
- …the techniques of the c.e. degrees and degrees in general seemed to have natural counterparts in the *LK* degrees.

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...until now

Main result

Theorem

If X, Y are Δ^0_2 and not low for K, then there exists a c.e. set A which is not low for K, such that $A \leq_{LK} X$ and $A \leq_{LK} Y$.

Answer to a question of Nies and Simpson (2006)

Corollary

If X, Y are relativly 1-random, this does not imply that the degrees LK-below both of them are K-trivial.

Contrast: This holds for LK replaced by Turing.

Corollary: Σ^0_1, Δ^0_2 structures

The Σ^0_1, Δ^0_2 structures of the LK degrees and the Turing degrees are not elementarily equivalent.

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Miller (2007) proved. . .

There are minimal pairs of LK, even in Δ^0_3.

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Theorem (Barmpalias/Lewis/Ng 2008)

There is a minimal pair for $\leq_{LK}$ which is $\leq_{LK} \emptyset'$.

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Proof:

- There is a Π^0_1 class with no low for K members, with all members $\leq_{LK} \emptyset'$ (Barmpalias/Lewis/Stephan 2007)
- Every Π^0_1 class contains a path with countably many LK-predecessors (Miller, Reimann/Slaman)
- Hence, by Jockusch-Soare methods, every Π^0_1 class contains a minimal pair for $\leq_{LK}$.

Structures $\leq_{LK} \emptyset'$ and $\leq_T \emptyset'$

Corollary

The structure of $\leq_{LK}$ restricted to $\leq_{LK} \emptyset'$ is not elementarily equivalent to $\leq_T$ restricted in Δ^0_2 or Σ^0_1.

Many questions remain

Much of the basic machinery for the study of this structure has been developed.

- Are the LK degrees an upper semi-lattice?
- Are the c.e. LK degrees dense?
- Is there a minimal LK degree?
- Characterize the LK degrees with countable lower cones.
- …and so on…see literature.

References

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Thank you!