

Randomness, Lowness and Domination

Work on progress

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MATHLOGAPS
Leeds, August 2006

Cantor Space, Topology and Measure

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- We use a standard Lebesgue measure on the Cantor space: the measure of a basic open set $[\sigma]$ is

$$\mu([\sigma]) = 2^{-|\sigma|}$$

This uniquely determines the Lebesgue measure of sets in 2^ω .

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Definition

A Martin-Löf test \mathcal{M} is a uniform sequence (E_i) of c.e. sets of binary strings such that $\mu(E_i) \leq 2^{-i}$. A real α avoids \mathcal{M} if some for i , $\alpha \notin E_i \Sigma^\omega$. A real number is called random if it avoids all Martin-Löf tests. W.l.o.g. assume $E_{i+1} \subset E_i$.

Other approaches to Randomness

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2. The **uncompressibility paradigm**: In random strings the ratio of information coded in initial segments and the length of these segments is maximal. Here the ratio is important and not the individual components. For example, there are random strings with low overall information content.

Examples

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- $\sum_e \sum_{\sigma \in V_e} 2^{-(|\sigma|+e)}$ where (V_e) is an effective enumeration of all c.e. prefix-free sets of strings.

Π_1^0 Classes and Randomness

Definition

A Σ_1^0 class is a collection C of reals such that

$$C = \{\alpha : \exists n(R(\alpha \upharpoonright n))\},$$

where R is a computable relation. Equivalently,

$$C = \{[\sigma] : \sigma \in W\},$$

where $W \subseteq 2^{<\omega}$ is c.e. (identify sets of sets of reals with their union).

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- Π_1^0 classes are the complements of Σ_1^0 classes.

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- If (E_i) is a Martin-Löf test, E_i is a Σ_1^0 class
- The random numbers is a Σ_2^0 class of measure 1, a countable union of Π_1^0 classes of measure tending to 1.

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- Relativizing we get: A is *low for random w.r.t. B* ($A \leq_{LR} B$) if every B -random is A -random.
- \leq_{LR} is transitive, Σ_3^0 and it contains \leq_T .

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- For some member U^A of a universal Martin-Löf test relative to A there is $V^B \in \Sigma_1^{0,B}$ with $\mu V^B < 1$ and

$$U^A \subseteq V^B$$

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- There is a least degree containing the low for random reals. Some of them are not computable (Kučera).
- It is not known if there is a least upper bound for any two degrees.
- The usual $A \oplus B = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in B\}$ is not a supremum (Nies).

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- There are c.e. sets which compute an a.e. dominating function but don't compute the halting problem (Cholak and others).
- The LR degree of the halting problem contains halves of minimal pairs in the c.e. Turing degrees (B. and Montalban).

A non-cuppable a.e. dominating c.e. degree

Theorem

There is an a.e. dominating function f (of c.e. Turing degree) such that for every c.e. A which does not compute the halting problem, $f \oplus A$ cannot compute the halting problem.

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Corollary

If A is c.e. and is computed by every a.e. dominating function of c.e. Turing degree then A is non-cuppable in the c.e. degrees.

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- strategies need better and better approximations of the measure of the domain of functional(s)
- only minimal restraints are allowed on the dominating function (at most one at each level of the tree)

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- How about the local structure of c.e. LR degrees? (density, minimal pairs etc.)
- Global constructions: minimal LR -degrees?

End

Thank you!