PA degrees, Π^0_1 classes and relative randomness

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Random degrees

- A number of results indicate that the complete randoms are very different to the incomplete randoms.
- In particular, that incomplete randoms have little information content.
- The theory of PA degrees was developed in parallel to the theory of random degrees in the early papers of Kučera.
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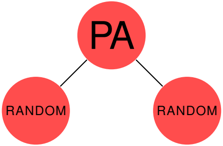
Joins of random degrees

Theorem (Barmpalias and Lewis)

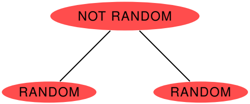
Every PA degree is the join of two random degrees.



Joins of random degrees

Corollary

Incomplete PA degrees give examples of pairs of random degrees whose join is not random.



Proof idea

Let C be of PA degree. We wish to find randoms A, B such that $C \equiv_T A \oplus B$.

- We would like to start with a Π^0_1 class P of randoms and find A, B inside P,
- using the fact that C computes a member of every nonempty Π^0_1 class.
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Trees

If T is a partial function from $2^{<\omega}$ to $2^{<\omega}$ we say that T is a tree if for every $\sigma \in 2^{<\omega}$ and $i \in \{0, 1\}$ such that $T(\sigma * i)\downarrow$:

- $T(\sigma)\downarrow \subset T(\sigma * i)$;
- $T(\sigma * (1 - i))\downarrow | T(\sigma * i)$.

A tree T is perfect if $T(\sigma)\downarrow$ for all σ. A finite tree T of level n is the restriction of a tree (as a map) to strings of length n.

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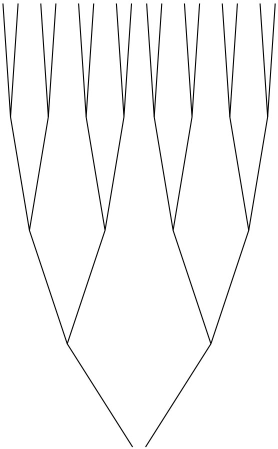

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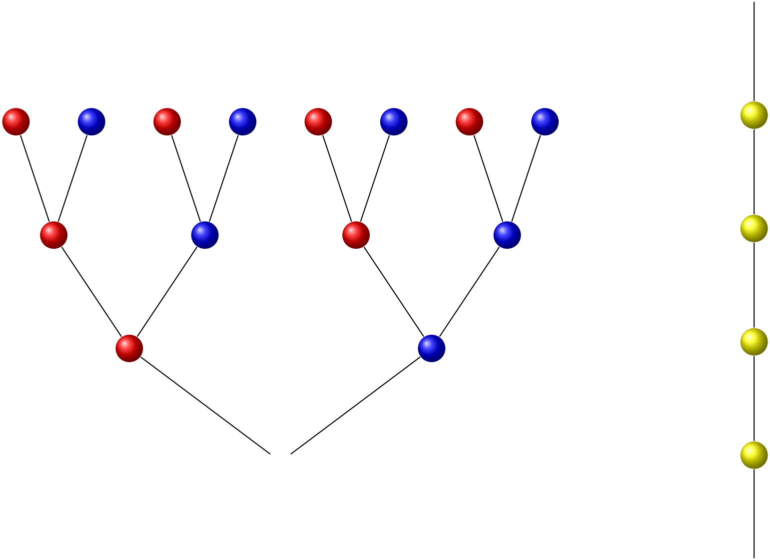

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- We could use this tree to define A, B in P such that $A \oplus B \geq_T C$:
- Get A from X by making the nth special position equal to $C(n)$
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The plan...

- Then we would only need to find such a homogenous tree which is computable from C.
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- However the class of trees of this type does not form a Π^0_1 class!
- In fact, the topological space $\mathcal{T}$ of perfect trees (or even trees of this type) is not compact, hence not homeomorphic to the Cantor space.
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A way out . . .

- We work inside a compact subspace of $\mathcal{T}$.
- Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function
- Function f defines a partition on any given infinite string A.
- Let $(\sigma_A(i))$ be the unique sequence of strings such that $|\sigma_A(i)| = f(i)$ and

$$A = \sigma_A(0) * \sigma_A(1) * \ldots \qquad (1)$$

- Say that A, B are *piecewise f-different from level n* if $\sigma_A(i) \neq \sigma_B(i)$ for all $i \geq n$.

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The subspace

- For any such pair A, B define the tree $T^{f,n}_{AB}$ as follows:

$$\begin{aligned} T^{f,n}_{AB}(\emptyset) &= A \restriction f(n) \\ T^{f,n}_{AB}(\tau * 0) &= T^{f,n}_{AB}(\tau) * \min\{\sigma_A(|\tau|), \sigma_B(|\tau|)\} \\ T^{f,n}_{AB}(\tau * 1) &= T^{f,n}_{AB}(\tau) * \max\{\sigma_A(|\tau|), \sigma_B(|\tau|)\} \end{aligned}$$

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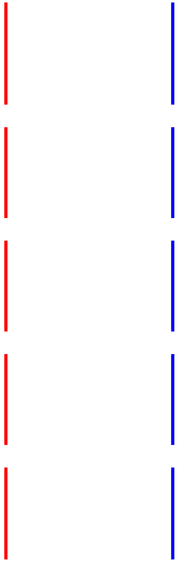

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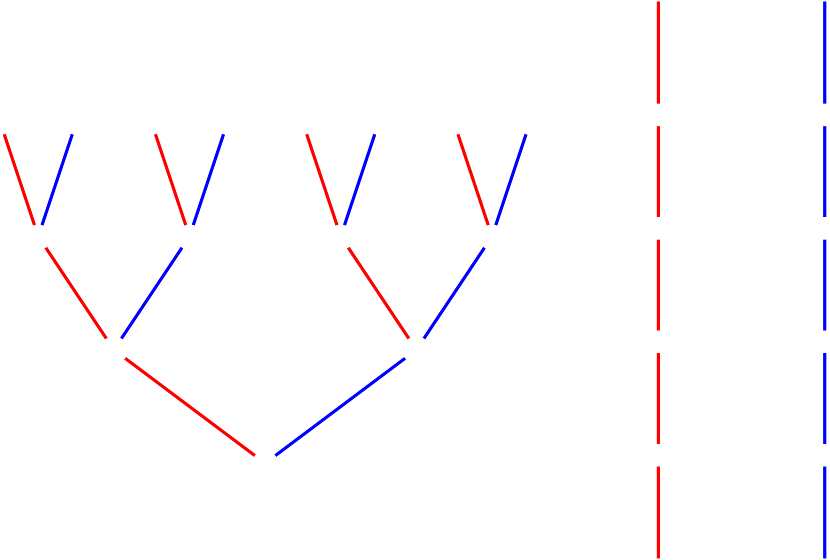

The subspace

- $\mathcal{T}^{f,n}$ is compact and f-effectively homeomorphic to the Cantor space.
- If f is recursive, the trees in $\mathcal{T}^{f,n}$ which are contained in the Π^0_1 class P is itself a Π^0_1 class!
- The set

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Theorem

There exists a computable function f such that if P is a Π^0_1 class and $X \in P$ is sufficiently random then for some $n \in \mathbb{N}$ and some Y piecewise f-different to X from level n we have $[T^{f,n}_{XY}] \subseteq P$.

Sketch of proof

- X can be *f-switched after level n inside P* if there is a Y piecewise f-different to X such that $[T^{f,n}_{XY}] \subseteq P$
- Let D_n be the reals that cannot be f-switched from level n inside P.
- $\hat{D}_n = D_n \cap P$
- We show by induction that $\mu(\hat{D}_n) \leq \mathcal{O}(2^{-n})$
- … for a recursively defined f
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More details

- We say that A can be *f-switched at $[n, m]$ inside P* if it belongs to P and there exists B such that $\sigma_A(i) \neq \sigma_B(i)$ for all $i \in [n, m]$ and for every sequence $(x_i) \in \{A, B\}^\omega$ with $x_i = A$ for $i \notin [n, m]$, the real $\sigma_{x_0}(0) * \sigma_{x_1}(1) * \ldots$ is in P.
- A real can be f-switched at $[n, m]$ for all $m \in \mathbb{N}$ inside P iff it can be f-switched from level n inside P.
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- Hence $D_n = \cup_m D_{n,m}$.
- the sequence $D_{n,m}$ is uniformly Σ^0_1 for recursive f.
- $\hat{D}_{n,m} = P \cap D_{n,m}$.
- It suffices to construct f such that

$$\begin{aligned} \mu(\hat{D}_{n,n}) &\leq 2^{-n-1} \\ \mu(\hat{D}_{n,m+1} - \hat{D}_{n,m}) &\leq 2^{-n-m-2} \end{aligned}$$

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Equivalence classes of Relative Randomness

- The *LR degrees* are equivalent classes containing oracles which induce the same notion of relative randomness.
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Any LR degree forms a minimal pair with almost all others. In particular every the LR degrees of two relatively 2-random sets form a minimal pair.

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Every nonempty Π^0_1 class contains a path which is low for Ω.

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Given a countable sequence (C_i) of sets such that $C_i \not\leq_{LR} \emptyset$ for all $i \in \mathbb{N}$, and a nonempty Π^0_1 class P, there exists $B \in P$ such that $C_i \not\leq_{LR} B$ for all i.

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Every nonempty Π^0_1 class contains two paths with greatest lower bound 0 in the LR degrees.

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Corollary (Barmpalias and Ng)

There is a minimal pair of LR degrees below $\mathbf{0}'$.

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- By Barmpalias, Lewis, Stephan there is a Π^0_1 class without K-trivial paths such that all of its paths are LR-below $\emptyset'$.
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More applications of Π^0_1 arguments to LR

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There exists a properly $\Delta^0_{\omega+1}$ LR degree below $\mathbf{0}'$.

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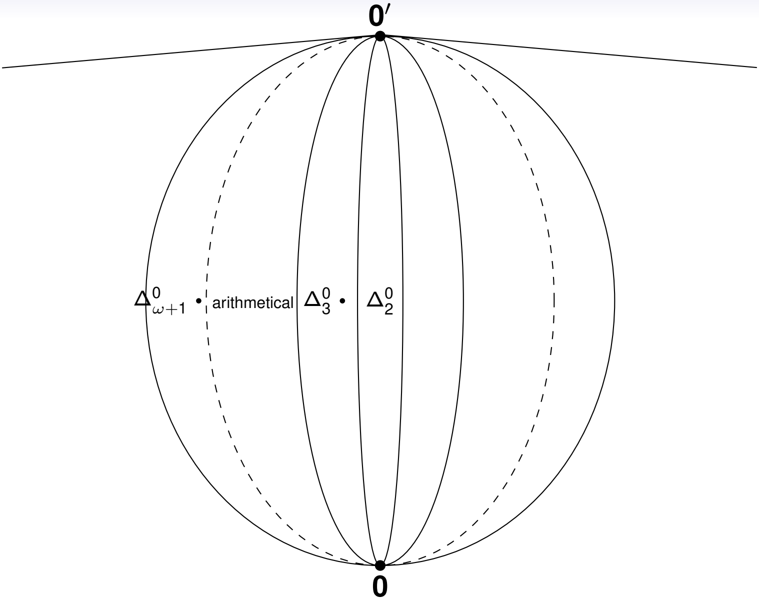
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