Computability and Applications to Analysis

Georgios Barmpalias

Submitted in accordance with the requirements for the degree of
Doctor of Philosophy

The University of Leeds
Department of Pure Mathematics

July 2004

Abstract

We study real numbers from the point of view of effectiveness and computability, especially regarding their approximations by 'effective' sequences of rational numbers. For this study we employ the two main classification methods from computability theory: hierarchies and degrees. We are especially interested in establishing connections with the classical theory. In chapter 1 we extend the hierarchy defined in Weihrauch and Zheng [33] (classifying the arithmetical reals) to cover all hyperarithmetical real numbers.

In chapter 2 we start the study of approximations of reals by means of degree structures. This is a new approach for the classification of the computably approximable (i.e. Δ_2) reals which yields a rich and interesting theory with many connections to the classical theory. To each computably approximable real x we assign a degree structure, *the structure of all possible ways available to approximate x.* We exhibit extreme cases of such approximation structures and prove a number of related results. In chapter 3 we continue the work of chapter 2 by studying further properties of the degrees of approximation representations (i.e. the elements of the approximation structure) of a real.

While the main issue of the previous two chapters was, given a real how rich (in a computational sense) can the variety of its representations be, chapter 4 deals with the reverse question: given an approximation representation A, how rich is the variety of reals which have approximation representation A? Furthermore, we start studying the structure of the wtt degrees which contain representations (of any real), as opposed to the study of the wtt degrees which contain representations of a fixed real.

In chapter 6 we give a characterisation of the approximation representations of computably enumerable reals as the sets which are both hypersimple and semi-computable (in the sense of Jockusch). Then we study the wtt degrees of hypersimple as well as hypersimple semicomputable sets. Chapter 5 is a note on algorithmic randomness.

Acknowledgements

I wish to thank Professor Barry Cooper for all the meetings and valuable discussions we had during the past few years. Also, the University of Thessaloniki, Greece and in particular the fund of Lamprini-Lina Athanassoula, as well as EPSRC for funding most of my PhD.

Contents

List of Figures

Introduction

We study real numbers from the point of view of effectiveness and computability, especially regarding their approximations by 'effective' sequences of rational numbers. For this study we employ the two main classification methods from computability theory: hierarchies and degrees. We are especially interested in establishing connections with the classical theory. In chapter 1 we extend the hierarchy defined in Weihrauch and Zheng [33] (classifying the arithmetical reals) to cover all hyperarithmetical real numbers. An intuitive idea is used for the definition which extends the 'finite case' of [33], but a characterisation of the related classes is obtained. The relationship with the classical hyperarithmetical hierarchy is established, as well as a hierarchy theorem and two fixed point theorems concerning computations related to the hierarchy. These hierarchies are defined by means of approximation properties of reals by effective sequences of rationals and turn out to differ from the classical arithmetical and hyperarithmetical hierarchies of sets (with respect to identification of sets A with reals $0.A$ with binary expansion the characteristic sequence of A).

In chapter 2 we start the study of approximations of reals by means of degree structures. This is a new approach for the classification of the computably approximable (i.e. Δ_2) reals which yields a rich and interesting theory with many connections to the classical theory. Although this approach naturally applies to all Δ_2 reals, we are most interested in the construction of computably enumerable reals and thus all the reals we construct in the rest of this thesis are such. To each computably approximable real x we assign a degree structure, *the structure of all possible ways available to approximate x.* This is defined by means of the weak truth table degrees of the *approximation representations* of x. So the main criterion for this classification is the variety of the effective ways we have to approximate a real number. We exhibit extreme cases of such approximation structures and prove a number of related results.

In chapter 3 we continue the work of chapter 2 by studying further properties of the degrees of approximation representations (i.e. the elements of the approximation structure) of a real. We show that the approximation structure is not necessarily dense and we exhibit two reals with the same information content but highly unrelated approximation structures. Finally we characterise the notion of approximation representation as the bi-infinite cut of a computable linear ordering of $\mathbb{N}$ of order type $\omega + \omega^*$.

While the main issue of the previous two chapters was, given a real how rich (in a computational sense) can the variety of its representations be, chapter 4 deals with the reverse question: given an approximation representation A, how rich is the variety of reals which have approximation representation A? Furthermore, we start studying the structure of the wtt degrees which contain representations (of any real), as opposed to the study of the wtt degrees which contain representations of a fixed real (which we did earlier). The importance of this structure for classical computability theory can be in the next chapter where we establish strong connections with classical notions.

Chapter 5 is a digression from the main theme of the thesis and is concerned with the notion of randomness in an algorithmic context. We examine the relation of randomness with two notions from classical computability theory: immunity and the difference hierarchy. We show that although every random set is ω-immune, no random set is hyperimmune. Moreover, we give a necessary condition for the class of f-c.e. sets to have random members; we deduce that if f is bounded by a polynomial, there are no f-c.e. random sets.

In chapter 6 we give a characterisation of the approximation representations of computably enumerable reals as the sets which are both hypersimple and semi-computable (in the sense of Jockusch). Then we study the wtt degrees of hypersimple as well as hypersimple semicomputable sets. We show that outside every non-trivial cone of Turing degrees there is a non-trivial hypersimple free cone of c.e. wtt degrees and that there are hypersimple semicomputable free cones of c.e. wtt degrees with hypersimple base. Next we show that the hypersimple free wtt degrees are downwards dense in the c.e. wtt degrees and that for any hypersimple wtt degree there is another one strictly above it. Then we turn to hypersimple semicomputable wtt degrees and show that there is no greatest such degree, while we leave open the question of existence of maximal such degrees. However we do construct two hypersimple semicomputable degrees such that every degree above them is not hypersimple semicomputable, which implies that this structure is not an upper semi-lattice.

Some joint work done in the last three years with X. Zheng and R. Rettinger in different aspects of computable analysis can be found in [35, 34]. Special care has been taken to make each chapter reasonably self-contained with respect to the rest of the thesis. However we assume some background in computability theory and especially relating to priority arguments (finite, infinite and tree arguments). For this we refer to [31, 23, 24]. Unexplained notation in this thesis is quite standard.

Chapter 1

A Transfinite Hierarchy of Reals

1.1 Introduction

A real number x can be represented by its binary expansion, i.e. a set A such that $n \in A \iff$ the n-th binary digit of x is 1. In this case we write $x = x[A]$. Thus the classical hierarchies of computability theory (e.g. the arithmetical, the hyperarithmetical hierarchy) can be seen as hierarchies of reals. However, the classes of reals we get in this way are not natural from the point of view of computable analysis [1]; for, important classes like c.e. or co-c.e. reals (that is, left approximable and right approximable reals, see [33]) are not classes of these hierarchies. Weihrauch and Zheng [33] defined a natural hierarchy of length ω which is closely related to the (recursion theoretic) arithmetical one *and* classes as the c.e. and co-c.e. reals are classes of the hierarchy[2]. Moreover the definition reflects the difficulty of approximating a real by a sequence of rationals. The purpose of this chapter is to extend this hierarchy as far as possible, by using the same idea: reals are classified according to the 'order' of the prefix of $sup - inf$ alternations that is needed in front of a computable object, in order to get them (this statement will become more clear and precise in the following sections). In section 1.2 we give the intuitive idea behind the definition given in the next section. In section 1.4 we give a characterization of the reals of a class in the hierarchy which allows us to prove the hierarchy theorem in section 1.5. In section 1.4 we also prove the invariance of the hierarchy under the system of notation used in its definition. The relation of our hierarchy to the hyperarithmetical one is given in section 1.6. Finally a couple of fixed

[1] This is only one disadvantage of the binary representation of reals. In general this representation is not acceptable in computable analysis because it gives rise to peculiar situations.

[2] other, lower complexity hierarchies have been studied in Rettinger, Zheng, Gengler, von Braunmhl[27], Weihrauch and Zheng[32] and Zheng, Weihrauch, Ambos-Spies[37].

point theorems regarding a special kind of computations with a variable oracle (which arose from the definition of the hierarchy) is given in the last section. The results of this chapter have been published in [2].

1.2 The basic idea

We want to extend the hierarchy defined in [33] in terms of finitely many alternations of *sup* and *inf* in front of a computable function $f : \mathbb{N} \to \mathbb{Q}$; and we want to use the same idea. Intuitively, at infinite successor levels (which correspond to infinite successor ordinals) we want to have an infinite prefix (and in particular of the order type of the corresponding ordinal) of *sup*-*inf* alternations. But of course, a computable function has only finitely many arguments, and so it is impossible to apply this idea directly. However, we can do it indirectly; in terms and notation of [33] we define $\Sigma_\omega = \Pi_\omega = \Delta_\omega$ to be the set of all arithmetical real numbers, i.e. $\cup_{i\in\omega}\Delta_i$. Now a number x in $\Sigma_{\omega+1}$ is one that can be obtained as the supremum of a sequence $\{x_i\}_{i\in\omega}$ whose terms are of the form

$$x_i = \sup_{i_1} \inf_{i_2} \ldots \Theta_{m(i)} f_i(i_1, \ldots, i_{m(i)}) \tag{1.1}$$

where Θ_i is $\sup_i$ if i is odd and $\inf_i$ otherwise; $\{f_i\}_{i\in\omega}$ is a computable sequence of computable functions with rational values such that f_i has $m(i)$ variables (for natural numbers), m computable (say non-decreasing, unbounded) function.

We can picture the computable functions (with rational image) arranged in a hierarchy of height ω so that at level 0 we have constants and at level $m > 0$ we have the computable functions of m arguments; and we regard a function f to be 'higher' than one g with less arguments because in general, f prefixed (with the usual $sup - inf$ prefix) can give more complex reals than g.

So, what we did in order to obtain a number in $\Sigma_{\omega+1}$ is to go *effectively* through the whole hierarchy of [33] (by choosing the 'rank' $m(i)$ and the program i of f and prefix it accordingly) and put a sup on the top. Intuitively, we picture x as

$$\sup \ldots \sup \inf F \tag{1.2}$$

where F is a 'higher-type' computable object, in this case a computable sequence of computable sequences of rationals; and speaking informally, the prefix has order-type $\omega + 1$ (reading it from right to left).

We can define $\Pi_{\omega+1}$ accordingly (by exchanging the occurrences of inf and sup in the above). At the next successor levels, we just increase number of arguments of the higher-type object, and add the usual $sup - inf$ prefix for those arguments. In particular, to get a number in $\Pi_{\omega+2}$ we do what we did in the case of $\Sigma_{\omega+1}$ but using a computable double sequence of computable sequences and fixing the second argument; so we get a sequence which goes effectively through $\Sigma_{\omega+1}$ whose infimum (if it exists) is a number in $\Pi_{\omega+2}$ (one can show that if we take the supremum of such a sequence, we will never get out of $\Sigma_{\omega+1}$). We picture such a number x as follows:

$$x = \inf \sup \ldots \sup \inf F$$

where this time F is a 'higher-type' computable object of order $\omega + 2$. We can continue in the way described above, and define even higher classes. What we need if we are on a given level is to be able to select effectively objects of lower levels, i.e. to compute the level and the object of that level we want to select. This requirement restricts us in the initial segment of computable (i.e. recursive) ordinals and also forces us to employ a system of notations for these ordinals; Kleene's $\mathcal{O}$ is a straightforward choice.

In the following, we often use the term 'finite limit':

Definition 1. *If we have an expression of the form $x(\boldsymbol{m}) = \lim_{i\to\infty} f(i, \boldsymbol{m})$ where $f : \mathbb{N}^{n+1} \to \mathbb{R}$, $n \geq 0$ and for all $\boldsymbol{m} \in \mathbb{N}^n$ $\lambda i.f(i, \boldsymbol{m})$ is eventually constant, then we say that x is a finite limit of f (or just a finite limit if it is clear which function is f or even that the limits in that expression are finite). We use the analogous expressions when we have* sup *or* inf *in place of* lim.

In order to make things simpler, we use the following

Proposition 1. *From $n > 0$ and a function $f : \mathbb{N}^n \to \mathbb{Q}$ such that*

$$x := \sup_{i_1} \inf_{i_2} \ldots \Theta_{i_n} f(i_1, \ldots, i_n) \textit{exists},$$

we can go effectively (in fact primitive recursively) to a function $g : \mathbb{N}^n \to \mathbb{Q}$ such that:

$$x = \sup_{i_1} \lim_{i_2} \ldots \lim_{i_n} g(i_1, \ldots, i_n)$$

where the limits are finite and the sequence $y_i = \lim_{i_2} \ldots \lim_{i_3} g(i_1, \ldots, i_n)$ is strictly increasing.

The proof of this can be obtained by iterating the procedure used in the proof of Lemma 3.1 in [33], and so it is omitted. With proposition 1 in mind we can think of expressions of the form $\sup\inf\ldots\Theta f$ as $\sup\lim\ldots\lim f$ and even expressions like (1.2) as $\sup\ldots\lim\lim F$ and so on; since when we have an effective sequence of $\sup\inf\ldots\Theta f$ expressions (as this was discussed above) we can transform it in a computable way to the $\sup\lim\ldots\lim f$ form.

1.3 The definition

After this informal discussion and in order to define the hierarchy of reals, we give a definition of the class of higher-type objects $\mathtt{HT}$ dependent on a system of ordinal notations $\mathcal{S}$. These computable objects ξ will be projected to $\mathbb{R}$ via suitable operators $\mathtt{Sup}, \mathtt{Inf}, \mathtt{Lim}$ and the expressions $\mathtt{Sup}\xi, \mathtt{Inf}\xi, \mathtt{Lim}\xi$ will represent what we informally called "prefix of an infinite order type over a higher type object".

Notation. Let $\mathcal{S}$ be a system of notation. We denote $|n|_{\mathcal{S}}$ the ordinal with notation n, and $p_{\mathcal{S}}$ the partial computable function which gives a notation for the predecessor of the ordinal represented by its argument (if this exists). We write $q_{\mathcal{S}}$ for the function which gives an index of a computable function whose successive values are notations for an increasing sequence of ordinals converging to the ordinal represented by its argument. $\square$

A system of notation $\mathcal{S}$ can be viewed as a well-founded tree: we say that $n <_{\mathcal{S}} m$ when $n, m \in \mathcal{S}$ and n can be obtained by applying successively the following operations starting from m:

- when we have $t \in \mathcal{S}$ which denotes a successor ordinal, we can apply $p_{\mathcal{S}}$ and get a new number
- when we have $s \in \mathcal{S}$ which denotes a limit ordinal, we can apply $q_{\mathcal{S}}$ and get any member of the resulting sequence.

One can see that $\langle \mathcal{S}, <_{\mathcal{S}} \rangle$ is a well-founded tree [3].

Lemma 1. *Let $\langle \mathcal{S}, <_{\mathcal{S}} \rangle$ be a notation system. There is a partial computable function f such that if $x \in \mathcal{S}$ then*

$$W_{f(x)} = \{y : y \leq_{\mathcal{S}} x\}$$

[3] If one takes Kleene's $\mathcal{O}$ then the order relation described here is identical to $<_{\mathcal{O}}$.

Proof. Enumerate in stages:

- *stage* 0: Enumerate x.
- *stage* $s+1$: Look at each member y of the finite set of elements already enumerated: if y is successor then enumerate $p_{\mathcal{S}}(y)$; and if limit, enumerate the first s elements of the sequence with index $q_{\mathcal{S}}(y)$.

It is not hard to prove that the algorithm works (if $y <_{\mathcal{S}} x$ is not enumerated, then prove by induction up to x that no z with $y <_{\mathcal{S}} z <_{\mathcal{S}} x$ is enumerated; a contradiction). □

Lemma 2. *Given a notation system $\langle \mathcal{S}, <_{\mathcal{S}} \rangle$ there is a partial computable function suc (called successor function) such that*

$$n <_{\mathcal{S}} m \Rightarrow suc(n,m) \leq_{\mathcal{S}} m \ \&\ p_{\mathcal{S}}(suc(n,m)) = n$$

Proof. To compute $suc(n,m)$ start enumerating the predecessors of m checking simultaneously the successor elements x whether $p_{\mathcal{S}}(x) = n$. When you find such number x, output x. □

Notation. We use the lower case Greek letters $\alpha, \beta, \gamma, \delta$ to denote ordinals and (in general) m, n, t, s, i, x for naturals. Also, we assume an effective enumeration of the partial computable functions $\{\lambda_i\}$, a pairing function $\langle ., . \rangle$ and its inverses $(.)_1$, $(.)_2$. We denote n-vectors by **bold face** letters. Finally in a prefix $\sup_{i_1} \inf_{i_2} \ldots \Theta_{i_m}$, Θ denotes the m-th term of this alternating sequence of $\sup, \inf$ (similarly for $\inf_{i_1} \sup_{i_2} \ldots \Theta_{i_m}$). □

Definition 2. *Given a system of notation $\mathcal{S}$ we define a hierarchy of functions* $\mathtt{HT} = \cup_{\gamma<\delta} K^\gamma$ *(where δ is the first ordinal not having a notation under $\mathcal{S}$) and index them simultaneously by induction up to δ. Let $\{c_n\}_{n \in A}$ be an effective numbering of constants in $\mathbb{Q}$ (A is a computable set of natural numbers).*

- stage 0:*We set*

$$\xi_{\langle n,t \rangle} = c_n$$

for all $n \in A$ and t with $|t|_{\mathcal{S}} = 0$. We denote the set of all such indices $\langle n,t \rangle$ by I^0 and $K^0 = \{\xi_i : i \in I^0\}$.

- stage $\beta + m$ *(here β is limit or 0 and $m > 0$): We set*

$$\xi_{\langle n,t\rangle} : \omega^m \to K^\beta$$
$$\xi_{\langle n,t\rangle}(\boldsymbol{x}) = \xi_{\lambda_n(\boldsymbol{x})}$$

for $t \in \mathcal{S}$ and all indices n of computable functions λ_n of m variables such that

1. $\forall \boldsymbol{x}\ \lambda_n(\boldsymbol{x}) \in I^\beta$
2. $\forall \boldsymbol{i}\ (\lambda_n(\boldsymbol{i}))_2 <_{\mathcal{S}} t$
3. $|t|_{\mathcal{S}} = \beta + m$

We denote the set of all such indices $\langle n,t\rangle$ by $I^{\beta+m}$ and $K^{\beta+m} = \{\xi_i : i \in I^{\beta+m}\}$.

- stage α *(where α is limit): We set*

$$\xi_{\langle n,t\rangle} : \omega \to \cup_{\gamma<\alpha} K^\gamma$$
$$\xi_{\langle n,t\rangle}(x) = \xi_{\lambda_n(x)}$$

for $t \in \mathcal{S}$ and all indices n of computable functions λ_n of one variable such that

1. $\forall x\ \lambda_n(x) \in \cup_{\gamma<\alpha} I^\gamma$ *(and so, $\forall x\ |(\lambda_n(x))_2|_{\mathcal{S}} < \alpha$)*
2. $\sup_i |(\lambda_n(i))_2|_{\mathcal{S}} = \alpha$
3. $\forall i\ (\lambda_n(i))_2 <_{\mathcal{S}} t$
4. $|t|_{\mathcal{S}} = \alpha$

We denote the set of all such indices $\langle n,t\rangle$ by I^α and $K^\alpha = \{\xi_i : i \in I^\alpha\}$.
Finally, we set $\mathtt{HT} = \cup_{\gamma<\delta} K^\gamma$, $I = \cup_{\gamma<\delta} I^\gamma$ and we say that $\xi \in \mathtt{HT}$ has rank $|\xi| = \gamma$ *, if it was generated at stage γ (i.e. it has an index in I^γ).*

Note that for any object ξ we generate in the above definition, we simultaneously code the stage it was generated into its index. This comes from our requirement to be able to select not only a program of an object of lower rank, but also the rank of that object.

Definition 3. *We define operators* $\mathtt{Sup}, \mathtt{Inf}, \mathtt{Lim}, :\subseteq \mathtt{HT} \to \mathbb{R}$*.*

If $|\xi| = 0$ then

$$\mathtt{Sup}\ \xi = \xi; \quad \mathtt{Inf}\ \xi = \xi; \quad \mathtt{Lim}\ \xi = \xi$$

If $|\xi| = \beta + m$ *then*

$$\begin{aligned}
\mathtt{Sup}\xi &= \sup_{i_1} \inf_{i_2} \ldots \Theta_{i_m} \overbrace{\mathtt{Lim}\,\underbrace{\xi(\boldsymbol{i})}_{\in K^\beta}}^{\in \mathbb{R}} \\
\mathtt{Inf}\xi &= \inf_{i_1} \sup_{i_2} \ldots \Theta_{i_m} \overbrace{\mathtt{Lim}\,\underbrace{\xi(\boldsymbol{i})}_{\in K^\beta}}^{\in \mathbb{R}} \\
\mathtt{Lim}\xi &= \lim_{i_1} \lim_{i_2} \ldots \lim_{i_m} \overbrace{\mathtt{Lim}\,\underbrace{\xi(\boldsymbol{i})}_{\in K^\beta}}^{\in \mathbb{R}}
\end{aligned}$$

And if $|\xi| = \alpha$ *limit then*

$$\begin{aligned}
\mathtt{Sup}\xi &= \sup_{i} \overbrace{\mathtt{Lim}\quad\underbrace{\xi(i)}_{\in \cup_{\gamma < a} K^\gamma}}^{\in \mathbb{R}} \\
\mathtt{Inf}\xi &= \inf_{i} \overbrace{\mathtt{Lim}\quad\underbrace{\xi(i)}_{\in \cup_{\gamma < a} K^\gamma}}^{\in \mathbb{R}} \\
\mathtt{Lim}\xi &= \lim_{i} \overbrace{\mathtt{Lim}\quad\underbrace{\xi(i)}_{\in \cup_{\gamma < a} K^\gamma}}^{\in \mathbb{R}}
\end{aligned}$$

In the above equations (and more generally) we always assume that the sup, inf*, lim on the right-hand side exist (and the function to which they apply is total). And finally we define the* $\mathcal{S}$*-hierarchy of reals:*

$$\mathtt{Sup}^0 = \mathtt{Inf}^0 = \mathtt{Lim}^0 = \textit{ The computable numbers}$$

$$\begin{aligned}
\mathtt{Sup}^{\beta+1} &= \{\mathtt{Sup}\xi : \xi \in K^\beta\} \\
\mathtt{Inf}^{\beta+1} &= \{\mathtt{Inf}\xi : \xi \in K^\beta\} \\
\mathtt{Lim}^{\beta+1} &= \{\mathtt{Lim}\xi : \xi \in K^\beta\}
\end{aligned}$$

$$\mathtt{Sup}^{\alpha} = \cup_{\gamma<\alpha}\mathtt{Sup}^{\gamma}$$
$$\mathtt{Inf}^{\alpha} = \cup_{\gamma<\alpha}\mathtt{Inf}^{\gamma}$$
$$\mathtt{Lim}^{\alpha} = \cup_{\gamma<\alpha}\mathtt{Lim}^{\gamma}$$

where α is limit.

In a similar way we can define the classes $\mathtt{Sup}^{\gamma}{}_n, \mathtt{Inf}^{\gamma}{}_n, \mathtt{Lim}^{\gamma}{}_n$ of sequences of n arguments for arbitrary n and the above definition would be a special case for $n = 0$. It is not difficult to verify then the following

Proposition 2. *A sequence $\{x_{\boldsymbol{m}}\}$ is in $\mathtt{Sup}_{\mathtt{n}}^{\gamma+\mathtt{t}}$ if there is $\{y_{\boldsymbol{mk}}\} \in \mathtt{Sup}_{\mathtt{t+n}}^{\gamma}$ (or $\mathtt{Inf}_{\mathtt{t+n}}^{\gamma}$; or $\mathtt{Lim}_{\mathtt{t+n}}^{\gamma}$) such that*

$$x_{\boldsymbol{m}} = \sup_{k_1} \inf_{k_2} \dots \Theta_{k_t} y_{\boldsymbol{mk}}$$

The cases $\mathtt{Inf}_{\mathtt{n}}^{\gamma+\mathtt{t}}$ and $\mathtt{Lim}_{\mathtt{n}}^{\gamma+\mathtt{t}}$ are analogous.

1.4 Normal form of the hierarchy

In this section we give a characterization of the reals belonging to a class K^{β}. Then we prove that the hierarchy of reals is in fact independent of the system of notation $\mathcal{S}$ used in its definition. By Φ_e we mean the e-th partial computable functional with rational values and $\{H(n)\}_{n\in\mathcal{S}}$ is the family of sets defined as in the hyperarithmetical hierarchy (see [28]) but with $\mathcal{S}$ in place of $\mathcal{O}$. For reference we give the following

Definition 4. *For a system of notation $\mathcal{S}$ define the family of sets $\{H(n)\}_{n\in\mathcal{S}}$ as follows. For all $n \in \mathcal{S}$,*

$$H(n) = \begin{cases} \emptyset & , |n|_{\mathcal{S}} = 0 \\ (H(p_{\mathcal{S}}(n)))' & , |n|_{\mathcal{S}} \text{ successor} \\ \{\langle i,j\rangle \mid j <_{\mathcal{S}} n \wedge i \in H(j)\} & , |n|_{\mathcal{S}} \text{ limit} \end{cases}$$

Theorem 1. *For any $\xi_n \in \mathtt{HT}$ we can find uniformly in its index n, programs e_i,q_i and computable functions ν_i such that*

1. if $|\xi_n|$ is limit:

$$\begin{aligned} \mathtt{Sup}\ \xi_n &= \sup_x \Phi_{e_1}(H(\nu_1(x)); x) \\ \mathtt{Inf}\ \xi_n &= \inf_x \Phi_{e_2}(H(\nu_2(x)); x) \\ \mathtt{Lim}\ \xi_n &= \lim_x \Phi_{e_3}(H(\nu_3(x)); x) \end{aligned} \tag{1.3}$$

and

- $\forall x\ \nu_i(x) <_{\mathcal{S}} (n)_2$
- $|\xi_n| = \sup_x |\nu_i(x)|_{\mathcal{S}}$

2. if $|\xi_n|$ successor

$$\begin{aligned} \mathtt{Sup}\ \xi_n &= \sup_x \Phi_{q_1}(H((n)_2); x) \\ \mathtt{Inf}\ \xi_n &= \inf_x \Phi_{q_2}(H((n)_2); x) \\ \mathtt{Lim}\ \xi_n &= \lim_x \Phi_{q_3}(H((n)_2); x) \end{aligned} \tag{1.4}$$

provided that the expressions on the left part of the equations are defined. If $|\xi_n| = 0$ *then the index* e_i *is that of the constant* ξ_n*. Moreover in the above, the function under the sup is strictly increasing, and the function under the inf, strictly decreasing.*

In the proof of the theorem we use freely the basic facts for computable functions (as the smn theorem) and ordinal notations, as well as proposition 1. We also use the following

Lemma 3. *Suppose that*

- $f(x, y) = \Phi_e(H(\lambda(x, y)); x, y)$
- $\forall x, y\ \lambda(x, y) <_{\mathcal{S}} \lambda_*(x)$

Then, we can find uniformly in λ, λ_*, e *a program* e_1 *such that*

$$f(x, y) = \Phi_{e_1}(H(\lambda_*(x)); x, y)$$

Proof. From the definition of $\{H(n) : n \in \mathcal{S}\}$ it follows that we can find uniformly in x, y an algorithm for the reduction $H(x) \leq_T H(y)$ (assuming $x <_{\mathcal{S}} y$). Now e_1 says: take x, y as input to the program e but to any questions which may occur during the computation answer in the way $H(\lambda(x, y))$ would answer (by consulting $H(\lambda_*(x))$). $\square$

Proof of the theorem. First we give an algorithm which takes n and if $\mathtt{Lim}\xi_n(\boldsymbol{x})$ exists for all $\boldsymbol{x}$, then it outputs a program e and a computable function λ such that

$$\mathtt{Lim}\ \xi_n(\boldsymbol{x}) = \lim_y \Phi_e(H(\lambda(\boldsymbol{x})); \boldsymbol{x}, y)$$

and

1. *if* $|\xi_n|$ *is limit then*
   - $|\xi_n| = \sup_x |\lambda(x)|_{\mathcal{S}}$
   - $\forall x\ \lambda(x) <_{\mathcal{S}} (n)_2$
2. *and if* $|\xi_n|$ *is successor then*
   - $|\xi_n| = |b|_{\mathcal{S}} + m$
   - $\forall \boldsymbol{x}\ \lambda(\boldsymbol{x}) = b$

   for $m > 0$ and $b <_{\mathcal{S}} (n)_2$ ($|b|_{\mathcal{S}}$ is the maximum limit less than $|\xi_n|$).

Given $n \in I^\beta$ the algorithm at some point will call itself but at an $m \in I$ which is generated at an earlier stage according to definition (2), i.e. $m \in I^\gamma$ for some $\gamma < \beta$. Hence, after finitely many calls it will reach some $m \in I^0$, in which case it is very easy to give an answer.

The algorithm

Given n we see whether $\xi_n \in K^0$. If yes, then we find the desired e, λ. Otherwise, we check whether it belongs to a limit level K^α of HT or to a successor level $K^{\beta+m}$.

- *Case* K^α*:* By definition we have $\mathtt{Lim}\xi_n(x) = \mathtt{Lim}\xi_{\lambda_{(n)_1}(x)}$. Now we can find uniformly in n functions τ, m such that given x, $|(\lambda_{(n)_1}(x))_2|_{\mathcal{S}} = |\tau(x)|_{\mathcal{S}} + m(x)$ where $\tau(x) <_{\mathcal{S}} (\lambda_{(n)_1}(x))_2$ and $|\tau(x)|_{\mathcal{S}}$ limit, $m(x) \in \omega$. So we have

  $$\mathtt{Lim}\xi_n(x) = \lim_{i_1} \ldots \lim_{i_m} \mathtt{Lim}\xi_{\lambda_{(n)_1}(x)}(\boldsymbol{i})$$

  where $m = m(x)$. By applying the algorithm itself on the index $\lambda_{(n)_1}(x)$ we get a program e and a function ν_1 with the property

  $$\mathtt{Lim}\ \xi_{\lambda_{(n)_1}(x)}(\boldsymbol{i}) = \lim_t \Phi_e(H(\nu(\boldsymbol{i}, x)); x, \boldsymbol{i}, t)$$

  and

- if $m(x) = 0 : \forall \boldsymbol{i}\ \nu(\boldsymbol{i}, x) <_{\mathcal{S}} \tau(x)$ (and $\sup_{\boldsymbol{i}} |\nu(\boldsymbol{i}, x)|_{\mathcal{S}} = \tau(x)$)
- if $m(x) > 0 : \forall \boldsymbol{i}\ \nu(\boldsymbol{i}, x) = \tau(x)$

From e, τ and ν we can find (by lemma 3) a program e_1 such that

$$\Phi_e(H(\nu(\boldsymbol{i}, x)); x, \boldsymbol{i}, t) = \Phi_{e_1}(H(\tau(x)); x, \boldsymbol{i}, t)$$

and so we have

$$\mathtt{Lim}\xi_n(x) = \lim_{i_1} \ldots \lim_{i_m} \lim_t \Phi_{e_1}(H(\tau(x)); x, \boldsymbol{i}, t)$$

and we can assume that all the limits except the first one are finite (because by proposition 1 we can find from e_1 another program which does the same job *and* this requirement is fulfilled). Now we know from the proof of Shoenfield's lemma that from e_1 we can find e_2 such that

$$\lim_{i_1} \ldots \lim_{i_m} \lim_t \Phi_{e_1}(H(\tau(x)); x, \boldsymbol{i}, t) = \lim_i \Phi_{e_2}(H(\lambda(x)); x, i)$$

where λ takes x and applies $m = m(x)$ times the successor function along the path of $(n)_2$. We output e_2 and λ.

- *Case $K^{\beta+m}$:* By definition we have

$$\mathtt{Lim}\xi_n(\boldsymbol{i}) = \mathtt{Lim}\xi_{\lambda_{(n)_1}(\boldsymbol{i})} = \lim_x \mathtt{Lim}\ \xi_{\lambda_{(n)_1}(\boldsymbol{i})}(x)$$

Now we call the algorithm for $\lambda_{(n)_1}(\boldsymbol{i})$ and (uniformly in n) we get a program e and a function ν with properties as in case K^α and

$$\mathtt{Lim}\ \xi_{\lambda_{(n)_1}(\boldsymbol{i})}(x) = \lim_y \Phi_e(H(\nu(\boldsymbol{i}, x)); x, y, \boldsymbol{i})$$

Now find e_1, ν_* such that

$$\mathtt{Lim}\ \xi_n(\boldsymbol{i}) = \lim_x \lim_y \Phi_{e_1}(H(\nu_*(\boldsymbol{i}, x)); x, y, \boldsymbol{i}) = \lim_x \Phi_{e_1}(H(\nu_*(\boldsymbol{i}, x)); x, \boldsymbol{i})$$

and $\forall x, \boldsymbol{i}\ \nu_*(\boldsymbol{i}, x) <_{\mathcal{S}} (\lambda_{(n)_1})_2(\boldsymbol{i})$ (we can do this because the operator of proposition 1 is primitive recursive). In the same way as above, from $\lambda_{(n)_1}$, e_1 and ν_* we can find e_2 such that

$$\Phi_{e_1}(H(\nu_*(\boldsymbol{i},x));x,\boldsymbol{i}) = \Phi_{e_2}(H((\lambda_{(n)_1}(\boldsymbol{i}))_2);x,\boldsymbol{i})$$

and so

$$\mathtt{Lim}\ \xi_n(\boldsymbol{i}) = \lim_x \Phi_{e_2}(H((\lambda_{(n)_1}(\boldsymbol{i}))_2);x,\boldsymbol{i})$$

We output $e_2, (\lambda_{(n)_1})_2$.

One could think of the above algorithm as a recursion over the well-founded tree $\langle I, <_* \rangle$ where $n <_* m \iff (n)_2 <_{\mathcal{S}} (m)_2$. It is not difficult to prove by induction up to the least ordinal which does not receive an $\mathcal{S}$- notation, that the algorithm does its job.

To prove the theorem, given $n \in I$ we check whether $|(n)_2|_{\mathcal{S}}$ is 0, successor $\beta + m$ or limit α. The case 0 is trivial.

- *Case $\beta + m$:*

$$\mathtt{Sup}\xi_n = \sup_{x_1}\inf_{x_2}\ldots\Theta_{x_m}\mathtt{Lim}\xi_n(\boldsymbol{x}) = \sup_{x_1}\inf_{x_2}\ldots\Theta_{x_m}\lim_y \Phi_e(H(b);\boldsymbol{x},y)$$

  Now, as usual, we can assume that the $\sup, \inf, \lim$ are finite (except for the first one) and find e_1 such that

$$\sup_{x_1}\inf_{x_2}\ldots\Theta_{x_m}\lim_y \Phi_e(H(b);\boldsymbol{x},y) = \sup_x \Phi_{e_1}(H((n)_2);x)$$

  because from b applying m times the successor along $(n)_2$ we get $(n)_2$. We output e_1.

- *Case α:*

$$\mathtt{Sup}\xi_n = \sup_x \mathtt{Lim}\xi_n(x) = \sup_x \lim_y \Phi_e(H(\lambda(x));x,y) =$$
$$\sup_x \Phi_{e_1}(H(suc(\lambda(x),(n)_2));x)$$

  We output e_1 and λ_* (where $\lambda_*(x) = suc(\lambda(x),(n)_2)$).

The cases $\texttt{Inf}, \texttt{Lim}$ are similar. □

Note that any number of the form $\lim_x \Phi_q(H((n)_2);x)$ can be written as

$$\lim_x \Phi_e(H(\nu(x));x) \text{ (with } \nu(x) <_{\mathcal{S}} (n)_2)$$

and similarly with $\sup, \inf$.

Theorem 2. *The converse of theorem 1 holds, i.e. given a real of the form*

- $\sup_x \Phi_e(H(\nu(x));x)$
- $|m| = \sup_x |\nu(x)|_{\mathcal{S}} = limit$
- $\forall x \ \nu(x) <_{\mathcal{S}} m$

we can find uniformly in e , ν, a program n such that

$$\sup_x \Phi_e(H(\nu(x));x) = \texttt{Sup}\xi_{\langle n,m\rangle}$$

and given a real of the form

- $\sup_x \Phi_e(H(m);x)$
- $m \in \mathcal{S}$
- $|m|_{\mathcal{S}}$ *successor*

we can find uniformly in e and m an index n such that

$$\sup_x \Phi_e(H(m);x) = \texttt{Sup}\xi_n$$

An analogous result holds for the cases of $\texttt{Inf}, \texttt{Lim}$*.*

Proof. An algorithm is needed similar to the one of the proof of theorem 1 but doing the converse job. The details are omitted. □

The above two theorems give a characterization of the real numbers which belong to a class (e.g. $\texttt{Sup}^\alpha$) of the hierarchy in terms of $\sup, \inf, \lim$ of a function $f : \mathbb{N} \to \mathbb{Q}$.

When we say that a $\xi \in \texttt{HT}$ is on a particular branch (of $\langle I, <_* \rangle$) we mean that its index lies on that branch. The following question arises: suppose we are given two branches of $\langle I, <_* \rangle$ of the same length. Then, would the corresponding classes of $\sup \xi$ for ξ on the one or the other branch differ? The following theorem says no.

Theorem 3. *Suppose we are given n and m lying on $\langle I, <_*\rangle$ such that $|(n)_2|_{\mathcal{S}} = |(m)_2|_{\mathcal{S}}$. We can find uniformly in n, m a program e and a function ν such that if* $\mathtt{Sup}\xi_n$ *is defined, then*

1. *if $|\xi_n|$ limit*
   - $\mathtt{Sup}\xi_n = \sup_x \Phi_e(H(\nu(x)); x)$
   - $\forall x\ \nu(x) <_{\mathcal{S}} (m)_2$
2. *if $|\xi_n|$ successor*

$$\mathtt{Sup}\xi_n = \sup_x \Phi_e(H((m)_2); x)$$

Similarly for $\mathtt{Inf}, \mathtt{Lim}$.

In the proof we use implicitly some lemmas which where originally proved (by Spector) for the case of $\mathcal{S} = \mathcal{O}$ (see [28]) but the same proofs work for an arbitrary system $\mathcal{S}$ (by replacing $<_{\mathcal{O}}$ with $<_{\mathcal{S}}$).

Lemma 4.
$$[x \in \mathcal{S}\ \&\ y \in \mathcal{S}\ \&\ |x|_{\mathcal{S}} = |y|_{\mathcal{S}}] \Longrightarrow H(x) \leq_T H(y),$$

uniformly in x and y. And

$$[x \in \mathcal{S}\ \&\ y \in \mathcal{S}\ \&\ |x|_{\mathcal{S}} = |y|_{\mathcal{S}} = successor] \Longrightarrow H(x) \equiv H(y),$$

uniformly in x and y.

Lemma 5. *For $x \in \mathcal{S}$*

$$\{u : u \in \mathcal{S}\ \&\ |u|_{\mathcal{S}} = |x|_{\mathcal{S}}\}$$

is recursive in $H(x)''$.

Proof of the theorem. If $|\xi_n|$ limit,
define $\nu(x) = suc(suc(\nu_1(x)))$ (ν_1 is from the theorem 1) and the program e says:

> start enumerating the $<_{\mathcal{S}}$ - predecessors of $(m)_2$ until you find the one x with $|x|_{\mathcal{S}} = |\nu_1(x)|_{\mathcal{S}}$. Now run the program e_1 of theorem 1 and to any questions that may occur, answer them in the way $H(\nu_1(x))$ would answer them (by consulting $H(\nu(x))$, as we have $H(\nu_1(x)) \leq_T H(\nu(x))$ uniformly in x).

If $|\xi_n|$ successor,

we can find uniformly in n, m a computable isomorphism for the equivalence

$$H((n)_2) \equiv H((m)_2).$$

The program e now says: take x and apply e_1 of theorem 1. To any questions that may occur, answer the way $H((n)_2)$ would answer by consulting $H((m)_2)$.
The cases $\mathtt{Inf}, \mathtt{Lim}$ are similar. □

Note that in the above theorem it is enough to have $|(n)_2|_{\mathcal{S}} \leq |(m)_2|_{\mathcal{S}}$ instead of $|(n)_2|_{\mathcal{S}} = |(m)_2|_{\mathcal{S}}$.

So far our hierarchy of reals is dependent on a fixed system of notation $\mathcal{S}$. And we saw that this hierarchy remains the same if we take as system of notation any branch of $\mathcal{S}$. So we only need to consider univalent systems of notation.[4] But it is well known that any univalent system of notation (i.e. a branch of a system of notation) is computably isomorphic to a branch of Kleene's $\mathcal{O}$. So, by combining the above results we get

Corollary 1. *The hierarchy of reals defined in definition 3 is independent of the system of notation $\mathcal{S}$ used in the sense that if $\mathcal{S}$ assigns notations to the ordinals up to $\beta \in On$ and $\mathcal{S}'$ up to $\alpha \in On$ with $\beta \leq \alpha$ then the first hierarchy is an initial segment of the second.*

1.5 The hierarchy theorem

Thus we have defined a unique hierarchy of reals which we get if we take $\mathcal{S}$ to be a maximal system of notation (i.e. one which assigns notations to all ordinals up to ω_1^{CK}). The next theorem asserts that the hierarchy never collapses.

Theorem 4. *For all $\alpha, \beta < \omega_1^{CK}$*

$$\alpha < \beta \Longrightarrow \begin{array}{l} \mathtt{Sup}^\alpha \subsetneq \mathtt{Sup}^\beta,\ \mathtt{Inf}^\beta,\ \mathtt{Lim}^\alpha \\ \mathtt{Inf}^\alpha \subsetneq \mathtt{Inf}^\beta,\ \mathtt{Sup}^\beta,\ \mathtt{Lim}^\alpha \\ \mathtt{Lim}^\alpha \subsetneq \mathtt{Lim}^\beta,\ \mathtt{Inf}^\beta,\ \mathtt{Sup}^\beta \end{array}$$

Definition 5. *In the following we write $f \leq_{wtt^*} \emptyset^\alpha$ if α is limit and*

[4] a univalent system of notation is one that assigns exactly one notation to every ordinal lying on an initial segment of the ordinals.

- $f(x) = \Phi(H(\lambda(x)); x)$
- $\sup_x |\lambda(x)|_{\mathcal{S}} = \alpha$
- $\forall x\ \lambda(x)$ *lie on a branch of a system of notation* $\mathcal{S}$

or if α *is successor and* $f \leq_T H(y)$ *for* $|y|_{\mathcal{S}} = \alpha$ *(*$\mathcal{S}$ *a system of notation).*

We know from the above that this definition is independent of $\mathcal{S}$.

Proof of theorem 4. The inclusions $\subseteq$ follow from theorem 1. We want to prove e.g. $\texttt{Sup}^{\alpha} \subset \texttt{Sup}^{\alpha+1}$ when α is limit. Assume that λ is a computable function whose successive values form a sequence of notations for ordinals tending to α. Its enough to define a diagonal sequence $\{x_s\} \leq_{wtt^*} \emptyset^{\alpha}$ which fulfils the requirements:

$$R_{\langle e,i\rangle}: \quad \left.\begin{array}{l}\Phi_e(H(\lambda(i)); s) \text{ total} \\ \& \text{ strictly decreasing}\end{array}\right\} \Longrightarrow x = \sup_s x_s \neq \inf_s \Phi_e(H(\lambda(i)); s)$$

and in particular we can make them differ at their $\langle e,i\rangle$-th digit. But such a number $\inf_s \Phi_e(H(\lambda(i)); s)$ will be of the form $y = 0.y_1y_2\ldots$ in binary expansion where

- $y_k = \lim_s \Phi_{e_*}(H(\lambda(i)); s, k)$
- the limit is finite with modulus of convergence not more than $k+2$
- all values of $\lambda s.\Phi_{e_*}(H(\lambda(i)); s, k)$ equal 0 or 1

So it is enough to find a sequence $\{t_s\} \leq_{wtt^*} \emptyset^{\alpha}$ such that

$$\left.\begin{array}{l}\forall s\ f_{\langle e,i\rangle}(s) := \Phi_e(H(\lambda(i)); s, \langle e,i\rangle) \in \{0,1\} \\ \forall s \geq \langle e,i\rangle + 2\ f_{\langle e,i\rangle}(s) = f_{\langle e,i\rangle}(\langle e,i\rangle + 2)\end{array}\right\} \Longrightarrow$$

$$t_{\langle e,i\rangle} = 1 - \lim_s \Phi_e(H(\lambda(i)); s, \langle e,i\rangle)$$

In that case our diagonal real would be $x = 0.t_1t_2\ldots$ (as binary expansion). To find $t_{\langle e,i\rangle}$ check whether $\Phi_e(H(\lambda(i)); \langle e,i\rangle + 2, \langle e,i\rangle)$ is defined. If not, put $t_{\langle e,i\rangle} = 0$. Otherwise put

$$t_{\langle e,i\rangle} = 1 - \Phi_e(H(\lambda(i)); \langle e,i\rangle + 2, \langle e,i\rangle)$$

It is now easy to see that our requirements are fulfilled and also that $\{t_s\} \leq_{wtt^*} \emptyset^{\alpha}$.

In the case of a successor ordinal $\beta + 1$ we have to diagonalize over all

$$\sup_x \Phi_e(H(t_0);x)\ (|t_0|_{\mathcal{S}} = \beta)$$

within $\mathtt{Sup}^{\beta+1}$ and the proof is similar (even easier).

Also, the rest of the cases are proved in the same way (for the case of $\mathtt{Lim}^\alpha \subsetneq \mathtt{Inf}^\alpha$ take the diagonal sequence which starts with $0.1111\ldots$ and its n-th term is $0.t_1t_2\ldots t_n111\ldots$). □

Theorem 5. *If $m > 0$ and β limit and computable then*

- $\mathtt{Sup}^{\beta+\mathtt{m}+\mathtt{1}} \cap \mathtt{Inf}^{\beta+\mathtt{m}+\mathtt{1}} = \mathtt{Lim}^{\beta+\mathtt{m}}$
- $\mathtt{Sup}^{\beta+\mathtt{1}} \cap \mathtt{Inf}^{\beta+\mathtt{1}} \supsetneq \mathtt{Lim}^{\beta}$
- $\mathtt{Sup}^{\beta} = \mathtt{Inf}^{\beta} = \mathtt{Lim}^{\beta}$

and for any computable ordinal α

$$\mathtt{Sup}^\alpha \cup \mathtt{Inf}^\alpha \subsetneq \mathtt{Sup}^{\alpha+1} \cap \mathtt{Inf}^{\alpha+1}$$

Proof. This proof is in a sense a relativization of the proof of Lemma 3.3 in [33]. It is not difficult to see that a number x in $\mathtt{Sup}^{\beta+\mathtt{m}+\mathtt{1}}$ can be written as

$$x = \sup_i \inf_j f_1(i,j)$$

with

- $f_1 \leq_{wtt*} \emptyset^{\beta+m-1}$
- $f_1(i,j) < f_1(i,j+1)$
- $\sup_j f_1(i,j) > \sup_j f_1(i+1,j)$

A dual statement holds for a number x in $\mathtt{Inf}^{\beta+\mathtt{m}+\mathtt{1}}$:

$$x = \inf_i \sup_j f_2(i,j)$$

with

- $f_2 \leq_{wtt*} \emptyset^{\beta+m-1}$
- $f_2(i,j) > f_2(i,j+1)$

- $\inf_j f_2(i,j) < \inf_j f_2(i+1,j)$

We have $\inf_j f_2(i,j) < x < \sup_j f_1(i,j)$. We define the following function:

$$e(i) = \mu j[f_2(i,j) < f_1(i,j)]$$

It is $e \leq_{wtt*} \emptyset^{\beta+m-1}$. Define

$$f(i) = f_2(i, e(i))$$

and we have $f \leq_{wtt*} \emptyset^{\beta+m-1}$. It is

$$\inf_j f_2(i,j) \leq f_2(i,e(i)) = f(i) < f_1(i,e(i)) \leq \sup_j f_1(i,j)$$

So $\lim_i f(i) = x$ which means that $x \in \mathtt{Lim}^{\beta+\mathtt{m}}$. To prove that $\mathtt{Sup}^{\beta+1} \cap \mathtt{Inf}^{\beta+1} \supsetneq \mathtt{Lim}^{\beta}$ it is enough to define a diagonal real as in the proof of theorem 4 (in the case of limit ordinal α). The rest of the theorem follows easily. □

1.6 Relation to the hyperarithmetical hierarchy

As mentioned in the introduction, a real number x can be seen as a function $\mathbb{N} \to \mathbb{N}$, e.g. the characteristic function of the set A which corresponds to its binary expansion ($n \in A \iff$ the n-th digit of x is 1). So, the hyperarithmetical hierarchy of classical computability theory also applies to reals and the question is how do these hierarchies relate to each other (they have the same height). The following theorem shows that the hierarchy defined in this chapter is wider than the hyperarithmetical hierarchy (but the two hierarchies contain the same class of reals).

Definition 6 (Kleene). *Define the hyperarithmetical hierarchy as follows* [5]. *Let $\beta = \gamma + m$ where γ is limit or 0 and $\gamma = |y|_{\mathcal{O}}$ for $y \in \mathcal{O}$. Define*

$$\Sigma^0_\beta = \Sigma^{H(y)}_m$$
$$\Pi^0_\beta = \Pi^{H(y)}_m$$
$$\Delta^0_\beta = \Sigma^0_\beta \cap \Pi^0_\beta$$

[5] usually the classes which correspond to limit levels below are not considered and are replaced by the corresponding classes of the next level. However we include them, as we did in the definition of our hierarchy, because they are important classes.

Lemma 6. *If a set is H-c.e. where H is a hyperarithmetical set of a limit level (i.e. $H = H(y)$, for $y \in \mathcal{O}, |y|_{\mathcal{O}}$ limit), then it is also enumerated by a function of the form:*

$$f(n) = \Phi(H(\lambda(n)); n)$$

where λ is any computable function such that

- $\forall n \; \lambda(n) <_{\mathcal{S}} y_1$
- $\sup_i |\lambda(i)|_{\mathcal{S}} = |y_1|_{\mathcal{O}}$

and $\mathcal{S}$ any system of notation which assigns notation to $|y|_{\mathcal{O}}$ and $|y_1|_{\mathcal{S}} = |y|_{\mathcal{O}}$.

Proof. Let $m_0 \in H$. Without loss of generality we assume $\mathcal{S} = \mathcal{O}$ and $\lambda <_{\mathcal{O}}$ - increasing. Suppose that A is enumerated by $\Phi(H; n)$. Note that $H = \{\langle i, j\rangle : i \in H(j) \wedge j <_{\mathcal{O}} y\}$. Now at stage s we enumerate the $<_{\mathcal{O}}$ - predecessors (which appear by the s-th stage of the particular enumeration) of $\lambda(0), \lambda(1), \ldots, \lambda(s)$, in a set D_s. Then we run the computation $\Phi(H; n)$ for each $n < s$ and if some question "$\langle i, j\rangle \in H$? " occurs we do the following: check whether $j \in D_s$. If not, output m_0 and forget this (unfinished) computation for this stage. If yes, then answer the question (which is really about whether "$i \in H(j)$?") by consulting $H(\lambda(s))$, and continue the computation doing the same thing, until the computation is finished (so you output the result) or cancelled because we are unable to answer a question (this is the case when $j \notin D_s$). After cancel or finish all computations for $n < s$, go to stage $s+1$. It is not difficult to see that the program we defined does its job (if equipped with the 'variable-oracle' $H(\lambda(n))$). □

Proposition 3. *(i) If β is successor then*

$$A \in \Sigma^0_\beta(\Pi^0_\beta) \underset{\not\Leftarrow}{\Rightarrow} x[A] \in \texttt{Sup}^\beta(\texttt{Inf}^\beta)$$

and if α is limit

$$A \in \Sigma^0_\alpha(=\Pi^0_\alpha) \iff x[A] \in \texttt{Sup}^\alpha(=\texttt{Inf}^\alpha)$$

and for any $\gamma < \omega_1^{CK}$

$$A \in \Delta^0_\gamma \iff x[A] \in \texttt{Lim}^\gamma$$

(ii)

$$A \in \Delta_1^1 \iff x[A] \in \cup_{\gamma<\omega_1^{CK}} \mathtt{Lim}^\gamma$$

Proof. The only interesting part is to find a set $A \notin \Sigma_\beta^0$ such that $x = x[A] \in \mathtt{Sup}^\beta$ (we can similarly do the dual case). Let $\beta = \gamma + 1$. Our requirements are:

$$R_e : x \neq 0.W_e^H$$

where $0.A = x[A]$ and H is a set of the hyperarithmetical hierarchy of level γ (i.e. $H = H(y)$ for some $y \in \mathcal{O}$ with $|y|_{\mathcal{O}} = \gamma$). This is because $A \in \Sigma_\beta^0 \iff A$ is c.e. in H. Now we keep the $2e + 1$ - th place in the decimal expansion of x, for the e-th requirement. We start with the rational $0.001010101\ldots$. At stage s we have an enumeration of W_i^H for $i < s$ and we output the rational we had in the previous stage with the following changes: We check for each $2i + 1 < s$ whether it belongs to the set of elements of $0.W_i$ so far enumerated. If yes, we put 0 in the $2i + 1$-th position and 1 in the $2i$ position. Our sequence is increasing, and its supremum x will satisfy all of our requirements. Moreover the sequence is of the form $\Phi(H(y); n)$, if γ successor, or $\Phi(H(\lambda(n)); n)$ (with $\forall n\ \lambda(n) <_{\mathcal{O}} y$ and $\sup_n |\lambda(n)|_{\mathcal{O}} = \gamma$) if γ limit (the last due to lemma 6) [6].

To prove $\Delta_\gamma^0 = \mathtt{Lim}^\gamma$ use Shoenfield's lemma and note that for any function of the form $f(n) = \Phi(H(y); n)$ where $|y|_{\mathcal{O}} = \gamma$ limit, there exists $f_* \leq_{wtt*} \emptyset^\gamma$ such that $\lim_n f(n) = \lim_i f_*(i)$. □

1.7 Fixed point theorems

In this section we prove two fixed point theorems regarding computation with a non-fixed oracle, i.e. of the form $f(x) = \Phi(H(\lambda(x)); x)$. Here, Φ denotes a computable functional with rational values.

Lemma 7. *(i) If A is a computable set and $\mathbb{N} - A$ does not contain a collection of indices for all partial computable functions , then every computable function f has a fixed point in A. Moreover, given A we can find it uniformly in f.* [7]

(ii) If $A \leq_T B$ and $\mathbb{N} - A$ does not contain a collection of indices for all partial computable functions , then every B- computable function f has a fixed point in A. Moreover, given A we can find it uniformly in B and f.

[6] the idea for this diagonalization is from Downey[12].

[7] From this, it follows easily that every partial computable function ψ with $Dom\psi$ containing the set of indices of a partial computable function, has a fixed point.

(iii) *The above (i), (ii) hold for functionals instead of functions: If $A \leq_T B$ and $\mathbb{N} - A$ does not contain a collection of indices for all partial computable functionals, then for every B- computable function f, given A we can find uniformly in B and f an $e \in A$ such that $\Phi_{\lambda(e)} \simeq \Phi_e$.*

Proof. (i) Take

$$f_*(x) = \begin{cases} f(x), & x \in A \\ x_0, & \text{otherwise} \end{cases}$$

where $\forall x \notin A\ \neg[\lambda_x \simeq \lambda_{x_0}]$ (and so $x_0 \in A$). Then choose b such that

$$\lambda_{f_*(\lambda_x(x))} \simeq \lambda_{\lambda_b(x)}$$

Now the fixed point $\lambda_b(b)$ is in A.

(ii) This is a relativization of (i).

(iii) The proof is similar to the above.

□

Theorem 6. *Suppose that $f(s,n) \simeq \Phi_e(H(\lambda_d(s)); s, n)$ for some indices e, d of partial computable functions. Then we can find $(\emptyset' \oplus <_{\mathcal{S}})'$ - uniformly in e, d an index e_* such that:*

$$f(s,n) \simeq \Phi_{e_*}(H(\lambda_{e_*}(s)); s, n)$$

Proof. Take

$$A = \{x : \forall n[\lambda_d(n) \downarrow \Longrightarrow \lambda_d(n) <_{\mathcal{S}} \lambda_x(n)]\}$$

A can be decided by a $(\emptyset' \oplus <_{\mathcal{S}})'$ -oracle. Obviously $\forall x \notin A$, $\neg[\lambda_d \simeq \lambda_x]$. Define (uniformly in e, d) a total function g such that:

$$\Phi_{g(x)}(H(\lambda_x(s)); s, n) \simeq \Phi_e(H(\lambda_d(s)); s, n)$$

whenever $\lambda_d(s) <_{\mathcal{S}} \lambda_x(s)$. By lemma 7 we can find $(\emptyset' \oplus <_{\mathcal{S}})'$- effectively an index $e_* \in A$ such that $\Phi_{g(e_*)} \simeq \Phi_{e_*}$.

So we have

$$\Phi_{e_*}(H(\lambda_e(s)); s, n) \simeq \Phi_e(H(\lambda_d(s)); s, n) \tag{1.5}$$

when $\lambda_d(s) <_{\mathcal{S}} \lambda_{e_*}(s)$. But since $e \in A$, (1.5) is true for all s. $\square$

Theorem 7. *Suppose that $f(s,n) \simeq \Phi_{\lambda_{e(s)}}(H(\lambda_d(s)); n)$ for some indices e, d of partial computable functions. Then we can find $(\emptyset' \oplus <_{\mathcal{S}})'$ - uniformly in e, d an index e_* such that:*

$$f(s,n) \simeq \Phi_{\lambda_{e_*}(s)}(H(\lambda_{e_*}(s)); n)$$

Proof. The only difference from the previous proof is that now we find g such that

$$\Phi_{\lambda_{g(x)}(s)}(H(\lambda_x(s)); n) \simeq \Phi_{\lambda_{e(s)}}(H(\lambda_d(s)); n)$$

$\square$

Chapter 2

The Approximation Structure of a Computably Approximable Real

2.1 Introduction

The real numbers which are limits of computable sequences of rationals, also called recursively approximable reals (r.a. for short) form one of the most important classes of non-computable reals. We prefer calling them *computably approximable* (c.a.) according to the change of terminology in computability theory (adopted by many researchers in the field). By a result of Ho[18] they coincide with the $0'$-computable numbers, i.e. those that can be computed with pre-assigned accuracy using the halting set as an oracle (see [18]). There has been a lot of effort in order to classify c.a. reals and the main criterion was the *difficulty* to approximate them. One of the most successful attempts for such classification is Solovay's structure of computably enumerable (c.e.) reals (an important subclass of c.a. reals) which really captures the notion of a c.e. real being more difficult to approximate from another. The maximal elements of this structure, intuitively being the hardest c.e. reals to approximate, turn out to be *random* reals (see [12]). However Solovay's approach is applied only to c.e. reals [1] although a number of more recent approaches (via reducibilities or hierarchies) deal with more general classes. For example Rettinger and Zheng[26] (see also [27]), define a dense hierarchy of c.a. numbers which transcends the c.e. reals (yet it does not exhaust the c.a. reals). The underlying idea of this classification is how 'slow' is the 'fastest' computable sequence with limit a

[1] of course it is dually applied to co-c.e. reals.

particular real (see Zheng[36] for a survey of results in this direction). Other approaches have to do with reducibilities e.g. Downey, Hirschfeldt and LaForte[13] where particular reducibilities are introduced as a measure of relative randomness.

In this chapter we present a different approach for classifying c.a. reals: our criterion is *the variety of the possible ways to approximate a real.* Using restricted oracle computations we make this statement precise: having a real x and an approximation $\lim_s z_s = x$ we consider the set

$$A_z = \{s \mid z_s < x\}$$

which we may assume is infinite and co-infinite. We regard these sets as a sort of 'representations' of x and we study their complexity (and how they relate to the complexity of x). In particular, we order the class $\mathcal{S}_x$ of all such sets (for possible approximations of x) with a strong reducibility $\leq_r$ (e.g. $\leq_{\text{wtt}}$, $\leq_m$ etc.$\tilde{)}$ and we get a degree structure $\mathcal{D}_x^{\text{r}}$. Each element of $\mathcal{D}_x^{\text{r}}$ represents a different way to approximate x in terms of the restricted oracle computation associated with $\leq_r$. Indeed, if $A_z \leq_r A_y$ for z, y approximations of x, then given restricted access to the oracle A_y (which contains the information of which terms of y lie on the left of x) we can extract the relevant information about the approximation z. So in a way, y is at least as good as z. [2] If $\mathcal{D}_x^{\text{r}}$ has a maximum element, then there is a best approximation. And if it is trivial, i.e. consists of a single degree, then we could say that all ways available to approximate x are quite similar (with respect to $\leq_r$).

In the following sections we will exhibit a variety of structures $\mathcal{D}_x^{\text{r}}$ (which turns out to be a substructure of the r-degrees inside the Turing degree of x). In fact, we construct a c.e. real x such that an infinite antichain is embedable in $\mathcal{D}_x^{\text{wtt}}$. In this case the approximation structure is quite rich and intuitively there are a lot of different ways to approximate x. In the other extreme we construct a c.e. non-computable real x such that $\mathcal{D}_x^{\text{m}}$ is trivial, i.e. it consists of a single element. Such constructions of structures $\mathcal{D}_x^{\text{r}}$ with desired properties are done on a special framework for priority injury. In particular, the proof of theorem 10 has several interesting special features. A notion of 'links' is defined which is central in the actual construction; the links are actively involved in the priority list and they behave as negative requirements. But since they

[2] We note that all A_z contain the same information about x for various z with $\lim z = x$ (see proposition 4). The difference may be that this information is *arranged in different way.* This is the case when $A_z \not\equiv_r A_w$ for $\lim z = \lim w = x$. If $A_z \not\leq_r A_w$, the information in A_z is so much rearranged from the point of view of A_w, that a strong oracle procedure (based on $\leq_r$) is not enough to decode A_z from A_w.

are created during the construction (in a way which is not predictable) one could say that negative requirements are generated in the course of the construction and special care has been taken in order to control them.

In section 2.5 we note that some strong reducibilities coincide if we restrict ourselves to the class $\mathcal{S}_x$ for a real x; these are m, bounded tt with one query (also called btt(1)) and the positive reducibility. Finally in section 2.6 we are looking at the immunity properties of the sets in $\mathcal{S}_x$ for a given real x. The motivation for this is that when a real is e.g. c.e., then its complexity intuitively depends on how *rough* the right dedekind cut of it is. Theorem 8 says that no matter how complex x is, we can always produce infinitely many rationals in a very small area of x in the right cut of it. So one may want to see how the complexity of x depends on the complexity of $\mathbb{N} - A_z$ (in case the last is non-trivial, i.e. infinite). When a set A is (h or hh-) immune, intuitively it is difficult to make correct guesses about elements in that set (in case of immunity the output of a machine is viewed as a sequence of such guesses; in h-immunity an element of a strong array is a guess and it is correct if it intersects A; and in hh-immunity the notion of 'guess' is even weaker, corresponding to weak arrays). In this sense one may hope to get different classes of c.e. reals (with different 'complexity') by changing the immunity requirements on the set $\mathbb{N} - A_z$. We show that this is imposible; namely this set is either computable or h-immune and *not* hh-immune. Similar results are obtained for co-c.e. reals and non semi-computable ones.

Here is a list of conventions we adopt in the rest of this chapter:

- The expression $\Phi(A) = B;\ \varphi$ means that the equality holds and the calls to the oracle A are bounded by φ.
- We assume a standard 1-1 pairing function $\langle .,. \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
- The mode $[s]$ after a parameter of a construction means that we consider the value of the parameter at (end of) the s-th stage of the construction. Also, parameters which are not explicitely re-defined at some stage of the construction are assumed to preserve the value they had in the previous stage.
- If $\varphi_e(n)[s] \downarrow$ then $n, e < s$.

All rational sequences in this chapter are computable sequences of rational numbers. They are often represented by z, w (when their terms are z_s, w_s) and we usually drop the subscripts in their limits (e.g. we write $\lim z$ for $\lim_s z_s$). The sequence (Φ_e, φ_e) is an effective enumeration of the computable functionals/functions and the symbol $\downarrow$ in front of a requirement or a parameter in a construction means that it is satisfied or defined respectively (the symbol $\uparrow$ indicates the opposite situation). Many arguments are accompanied with illustrations in order to make them more comprehensible. For backround and basic definitions in computable analysis we refer to Zheng[36], Dunlop and Pour-El[16], while Odifreddi[23, 24] cover the computability theory used in this chapter. The results in this chapter are published in [4].

2.2 The approximation structure.

In this section we are going to give the definition of the approximation structure of a c.a. real x. Consider all computable sequences of rationals $z = \{z_s\}$ with $\lim z = x$ and for each of them, the sets

$$\begin{aligned} A_z &= \{s \mid z_s < x\} \qquad (2.1) \\ B_z &= \{s \mid z_s > x\} \end{aligned}$$

In the following we always consider z so that A_z is infinite and co-infinite (the other case being trivial).

2.2.1 Basic fact.

The following theorem shows that such sequences always exist. We note that this follows from the proof of theorem 12; however we give a direct proof since the more complicated argument in theorem 12 is based in the simple idea of the proof we are going to present now (so reading this proof will help understanding the latter).

Theorem 8. *If x is a c.a. real then there is a computable sequence of rationals $z = \{z_s\}$ with limit x and A_z infinite and co-infinite.*

For the proof, it is easy to see that if x is computable the result holds. And if x is not semi-computable then every (computable) sequence z with limit x has A_z infinite and co-infinite. So the only interesting case is when x is non-computable and semi-computable, say c.e. (the other case being dual). We will show how from an increasing computable sequence of rationals with limit x one can effectively obtain a sequence satisfying the requirements of the theorem.

Suppose that $\lim_s x_s = x$, $\{x_s\}$ is strictly increasing and $|x - x_s| < \frac{1}{f(n)}$ for a function $f : \mathbb{N} \to \mathbb{N} - \{0\}$ which is of course non-computable and $\lim_n f(n) = \infty$. The idea of the construction is that we are able to make guesses about rationals which lie on the right of x as the following figure shows

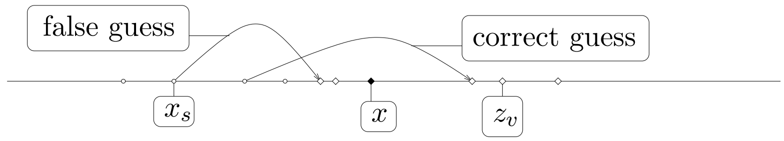

Figure 2.1: Guesses.

Suppose that at stage s we make a guess; if this is the n-th guess then we denote it (i.e. the rational which is proposed to be greater than x) by y_n^0. Because x is not co-c.e. (since it is not computable) it is hard to make correct (or, as we sometimes say, *successful*) guesses and some of the guesses will be false. The false guesses will be detected by the increasing sequence $\{x_s\}$; indeed, when at some stage s it is $x_s > y_n^0$ then we are sure that y_n^0 is on the left of x and so the guess is false. And this will definitely happen if the guess is false. When at stage s we discover a false guess (as above) we propose a correction y_n^1: this is the 1-st correction of the n-th guess. Later we may find (in the same way) that the correction itself is false, in which case we propose another correction. So for each guess y_n^0 we get a sequence of corrections $y_n^1, y_n^2, \ldots$ that will eventually reach a y_n^s which is indeed on the right of x. Moreover, we define the terms of a sequence $\{z_s\}$ to be the guesses and corrections produced during the construction and we ensure that $\lim_s z_s = x$ (by accumulating both the guesses and their corrections in smaller and smaller areas of x).

If at stage s the n-th guess has been made, we may also have some corrections by that time. So we define

$$y_n[s] = y_n^{t_0}$$

where $t_0 = \max\{t \mid \exists i \leq s \text{ with } z_i = y_n^t\}$. This is the most recent correction of the n-th guess (up to stage s) and we call it *the s-th version of the n-th guess*. We say that $y_n[s] \downarrow$ when $\{t \mid \exists i \leq s \text{ with } z_i = y_n^t\} \neq \emptyset$ (here by $z_i = y_n^t$ we don't mean just the equality but rather the *intentional* 'z_i was defined to be the t-th correction of the n-th guess'). Here is how we make the n-th guess at stage s:

$$y_n^0 = x_s + \frac{1}{s}$$

and any possible subsequent corrections of this guess will be

$$x_s + \frac{2}{s}, x_s + \frac{3}{s}, \dots$$

until we reach a rational greater than x. The last can be guaranteed if we choose x_0 such that $|x - x_0| < 1$. So for each term y_n^k we have

$$y_n^k = x_s + \frac{k+1}{s} \tag{2.2}$$

where s is the stage where y_n^0 was defined (i.e. the n-th guess was made). In the following when we say e.g. 'if $y_n[v] = x_s + \frac{k+1}{s} \dots$' (for some s, k) we don't mean just the arithmetical equation but rather 'if the v-th version of the n-th guess is its k-correction and the n-th guess was defined at stage $s \dots$'. At stage s a unique term will be defined, namely z_s. If it is defined via step A of the construction, then it is going to be a (new) guess; otherwise it is a correction of a previously made guess.

Construction.

Stage 0. Define $z_0 = x_0$.

Stage $s+1$. Two steps:

step A See whether $x_{s+1} > y_n[s]$ for any n with $y_n[s]\downarrow$. If not, then define

$$z_{s+1} = y_{n_0}^0 := x_{s+1} + \frac{1}{s+1} \tag{2.3}$$

where $n_0 = \mu t[y_t[s]\uparrow]$, and go to stage $s+2$. Otherwise go to step B.

step B Suppose that

$$\{n \mid x_{s+1} > y_n[s] \ \wedge \ y_n[s]\downarrow\} = \{i_k \mid k < m\}$$

(i_k distinct) and that

$$y_{i_k}[s] = x_{n_k} + \frac{t_k}{n_k} = y_{i_k}^{t_k - 1}.$$

for $k < m$. Then define

$$z_{s+k} = y_{i_k}^{t_k} := x_{n_k} + \frac{t_k + 1}{n_k} \tag{2.4}$$

for all $k < m$ and go to stage $s+m$.

About the construction.

1. In the definition (2.4) in step B of the construction, we regard z_{s+k}, $y_{i_k}[s]$ to be defined at stage $s+k$, for $k < m$.

2. In (2.3) the definition of n_0 means that there have been made $n_0 - 1$ guesses up to stage $s-1$ (so the next one is the n_0-th guess in the construction).

Verification

Lemma 8. *If y_m^0 was defined at stage s then $m < s$.*

Proof. By induction on m. For y_0^0 it holds since at stage 0 no guess is made. If it holds for all $i < m$ and y_m^0 is defined at stage s, then the $(m-1)$-th guess is already made by the end of stage $s-1$. So $m-1 < s-1$ which gives $m < s$. □

Lemma 9. *$\mathbb{N} - A_z$ is infinite.*

Proof. First we prove that every guess will eventually have a final correct version; formally

$$\forall n \; \exists s \; (y_n[s] > x).$$

Indeed, suppose otherwise. Then, according to the construction, there is an infinite sequence of corrections

$$y_n^0, y_n^1, y_n^2, \ldots$$

such that $y_n^t < x$ for all t. But

$$y_n^t = x_s + \frac{t+1}{s}$$

(where s is the stage where y_n^0 was defined) and since $x_0 + 1 > x$ and $x_0 < x_s$ for all s, we have $y_n^s = x_s + \frac{s+1}{s} > x$, a contradiction.

To complete the proof of the lemma, we show that for any n there is $s > n$ such that $z_s > x$. Indeed, at each stage s, exactly one term of $\{z_i\}$ is defined, namely z_s. Choose n; it is

$$z_{n+1} = y_k^t$$

for some t, k. According to the above, consider $t_0 > t$ such that $y_k^{t_0} > x$ and the stage s where $y_k^{t_0}$ was defined. It is $z_s > x$ and $s > n$, i.e. what we where looking for. □

The following lemma finishes the proof of the theorem.

Lemma 10. *$\lim_s z_s = x$.*

Proof. Choose $\epsilon > 0$; we will show that there is s_0 such that for all $s > s_0$ we have

$$|x - z_s| < \epsilon.$$

Choose n such that $\max\{\frac{1}{f(n)}, \frac{1}{n}\} < \epsilon$. Now choose s such that all i-guesses for $i \leq n$ have been successfully corrected. Formally, for all $s > s_0$ and $i \leq n$,

$$y_i[s] > x.$$

Consider $s > s_0$ and the term z_s. It will be

$$z_s = y_m[s] = y_m^{k-1} = x_t + \frac{k}{t}$$

for some t, k, m.

Claim. *$t > n$.*

Proof of claim. At stage s, z_s was defined either under step A or under step B. In the first case, $t = s > n$ (due to lemma 8). In the second case t is the stage where y_m^0 was defined and suppose that $t \leq n$ for a contradiction. Since $m < t$ by lemma 8, it is $m < n$. By the assumption we made about s_0, $y_m[s_0] > x$ and so the m-th guess (any version of it) is never considered in step B of any stage $v > s_0$, a contradiction. □

Claim. *If $z_s > x$ then we claim that*

$$|x - x_t - \frac{k}{t}| \leq \frac{1}{t}. \tag{2.5}$$

Proof of claim. Suppose otherwise for a contradiction, i.e. $|x - (x_t + \frac{k}{t})| > \frac{1}{t}$. Then

$$x < x_t + \frac{k-1}{t}$$

and since

$$y_m^{k-2} = x_t + \frac{k-1}{t}$$

the $(k-2)$-th version of the m-th guess (namely y_m^{k-2}) would be successful and y_m^{k-1} would never be defined, a contradiction. □

Since it is $t > n$, (2.5) gives $|x - z_s| < \frac{1}{n}$ and so $|x - z_s| < \epsilon$.

Now suppose that $z_s < x$ (it cannot be equal since x is not rational as it is non-computable). Then since $\{x_i\}$ is increasing and $t > n$ we have $x_t > x_n$ and so

$$|x - z_s| = |x - x_t - \frac{k}{t}| < |x - x_n - \frac{k}{t}| < |x - x_n| < \frac{1}{f(n)} < \epsilon.$$

which completes the proof. □

The theorem follows from the above lemmas.

just the theorem and proof of the

2.2.2 The definition.

One may want to consider the complement of the set A_z of (2.1); but since we assume it infinite and co-infinite, the two sets have roughly the same complexity which is directly related to the complexity of x (as we will see in the following). So it makes no difference which one we choose.

Each of these sets is a kind or 'representation' for x. We define a structure of all these representations (under a fixed reducibility) and we regard this as *the computability structure of the possible ways available to approximate the real x.* Fix a reducibility $\leq_r$ (e.g. T, wtt, tt, m etc.).

Definition 7. *Given a $0'$-computable real x, consider the class*

$$\mathcal{S}_x = \{A_z \mid \ z \textit{ computable and } \lim z = x\}$$

and the partially ordered set $\langle \mathcal{S}_x, \leq_r \rangle$. The elements of $\mathcal{S}_x$ are called x-sets. Also, consider the induced degree structure

$$\mathcal{D}_x^r = \{deg_r(A) \mid A \in \mathcal{S}_x\}.$$

This is called the approximation structure of x and its elements are called x-r-degrees.

Consider the case where $\lim z = \lim w = x$ for two computable sequences of rationals $z = \{z_s\}$, $w = \{w_s\}$. It is not difficult to prove that

Proposition 4. *If A_z, A_w are infinite and co-infinite then $A_z \equiv_T A_w \equiv_T x$.*

So $\mathcal{D}_x^{\mathrm{r}}$ is a substructure of the structure of r-degrees inside the Turing degree of x (see figure 2.2). Moreover, for any x, $\mathcal{D}_x^T$ is trivial, consisting of the Turing degree of x.

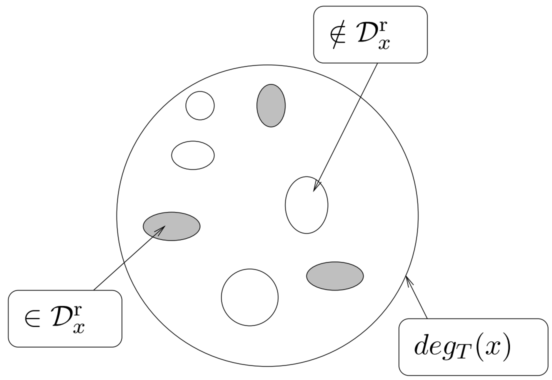

Figure 2.2: The approximation structure of x inside its Turing degree.

A natural question is whether this holds for stronger reducibilities. Not only this is not true, but we are able to construct reals with quite rich approximation structure (with respect to a strong reducibility). Due to our basic technique for such constructions, all reals constructed in this chapter will be c.e. In Barmpalias[3] we attempted a full approximation argument for such a construction, but the proofs in this chapter turn out to be simpler and establish much stronger results.

2.3 Antichain in $\mathcal{D}_x^{wtt}$.

We now construct a c.e. real x whose approximation structure is quite rich; namely an infinite antichain is embeddable in $\mathcal{D}_x^{\mathrm{r}}$.

Theorem 9. *There are c.e. reals x such that an antichain of wtt-degrees is embeddable in $\mathcal{D}_x^{wtt}$.*

For the proof, we are going to construct a sequence $\{z^n\}$ of computable sequences of rational numbers such that for each n, $\lim z^n = x$. Moreover, we will satisfy the following requirements:

$$R_{\langle e,i,j\rangle} : \neg[\Phi_e(A_{z^i}) = A_{z^j};\ \varphi_e]$$

(where $i \neq j$). In particular, at each stage s of the construction the requirement $R_{\langle e,i,j\rangle}$ will have a current witness $x_{\langle e,i,j\rangle}[s]$ and eventually we will succeed

$$\neg[\Phi_e(A_{z^i}; x_{\langle e,i,j\rangle}) = A_{z^j}(x_{\langle e,i,j\rangle});\ \varphi_e]$$

where $x_{\langle e,i,j\rangle}$ is the final witness of the requirement $R_{\langle e,i,j\rangle}$.

At each stage s we want the first s terms of the sequences $z^0, \dots z^s$ defined. It does not hurt if for the sequence z^n we define only the terms z^n_t for $t \geq n$ (since we can assume that e.g. $\forall t < n,\ z^n_t = 0$). So, at stage s we define the terms $z^0_s, \dots, z^s_s$.

The positive actions for R_t will be implemented via a non-decreasing sequence y which tends to x, the real we want to construct. At any stage s, the interval covered by y (namely $[0, y_s]$) is called *the black area* (see figure 2.5); and if a term enters the black area at some point, we call it a *black term*. In particular, when we want to put i into A_{z^j} at stage s (for the satisfaction of some requirement) we define $y_s = z^j_i$ (in other words z^j_i *enters the black area*). If at stage s no definition of y_s is mentioned, then we mean that y preserves its last value, i.e. $y_s = y_{s-1}$ and otherwise we say that y is *redefined* (at stage s); so we treat y as a parameter of the construction which changes values in the course of stages.

2.3.1 The definition of z^i_j

The terms of the sequences z^i are defined during the construction; the first thing we do at the begining of a stage is to define some more terms z^i_j.

At stage $s+1$ we divide the interval (y_s, w_s) where

$$w_s = \min\{z^n_i, 1 \mid n \leq i < s+1 \wedge z^n_i > y_s\} \tag{2.6}$$

into $s+3$ equal parts and set $z^0_{s+1}, \dots z^{s+1}_{s+1}$ on the borders (e.g. such that $z^0_{s+1} > z^1_{s+1} > \dots > z^{s+1}_{s+1}$); in other words

$$z^n_{s+1} = w_s - (n+1)\frac{w_s - y_s}{s+3} = y_s + (w_s - y_s)\frac{s+2-n}{s+3}$$

for all n with $0 \leq n \leq s+1$. So by stage s we have defined the terms z^i_j with $0 \leq i \leq j \leq s$, as figure 2.3 demonstrates.

2.3.2 The satisfaction of R_t

Now we describe the strategy for the satisfaction of R_t. This consists of two parts. So we break R_t into R^1_t and R^2_t; and when the first part has been completed, we put $R^1_t \downarrow$; and when the second part is completed we put $R^2_t \downarrow$ and the requirement R_t is satisfied. A few explanatory words are appropriate here. Suppose that $t = \langle e, i, j\rangle$. What we

Terms defined

$$
\text{\textit{Stages}}\quad
\begin{array}{|c|ccccc|}
\hline
0 & z_0^0 & & & & \\
1 & z_1^0 & z_1^1 & & & \\
2 & z_2^0 & z_2^1 & z_2^2 & & \\
\vdots & \vdots & \vdots & \cdots & \ddots & \\
s & z_s^0 & z_s^1 & \cdots & \cdots & z_s^s \\
\hline
\end{array}
$$

Figure 2.3: The terms defined by stage s.

really want to do is, having a current witness x_t for R_t, wait until $\Phi_e(A_{z^i}; x_t) \downarrow$ and if it is 0, put x_t into A_{z^j}—this is the action of R_t^2. But one can see that this may injure the computation as some elements *below the use* of the computation may enter A_{z^i} in the course of this action (i.e. by the redefinition of y). For this reason we must act *in advance*—act under R_t^1.

We must also keep some priority on the injuries, so after any action motivated by some requirement, say R_t, we *initialise* all requirements of lower priority (i.e. $R_n, n > t$) according to the following

Definition 8. *To initialise all $R_n, n > t$ at stage s, means to set*

$$x_{t+k} = s + k$$

for $k = 1, 2, \ldots,$ and $R_n^1 \uparrow$, $R_n^2 \uparrow$ for all $n > t$.

We note the following

Fact 9.1. *At any stage s and for any terms $z_{j_1}^{i_1}, z_{j_2}^{i_2}$ (already defined at s) which do not lie in the black area (i.e. $> y_s$) it is*

$$z_{j_1}^{i_1} > z_{j_2}^{i_2} \iff j_1 < j_2 \vee [j_1 = j_2 \wedge i_1 < i_2]$$

This follows from the way we define the terms z_j^i and will be proved in the verification as a lemma.

The action of R_t^1

We wait until $\varphi_e(x_t)\downarrow$ and suppose that this happens at stage s. If there are $z_k^i < z_{x_t}^j$ (so $k > x_t$, by fact 9.1) not in the black area (i.e. $z_k^i > y_{s-1}$), with $k < \varphi_e(x_t)$ then put

$$y_s = \max\{z_k^i \mid k \leq s \wedge z_k^i > y_{s-1} \wedge z_k^i < z_{x_t}^j\}$$

and $R_t^1\downarrow$ (remember that $t = \langle e, i, j\rangle$). Also we *initialise* all R_n, $n > t$. When such an action is performed we say that R_t^1 *receives attention.*

The action of R_t^2

When we know that R_t^1 has acted (that is when $R_t^1\downarrow$), then we draw our attention to the satisfaction of R_t^2: we wait until $\Phi_e(A_{z^i}; x_t)\downarrow$ and

1. If $\Phi_e(A_{z^i}; x_t) = 0$ *and* the use of the computation is below $\varphi_e(x_t)$ then we define

$$y_s = z_{x_t}^j$$

   thus putting x_t into A_{z^j}.

2. Initialise all R_n for $n > t$ and set $R_t^2\downarrow$.

When this action is performed we say that R_t^2 *receives attention.*

More about the construction

We say that R_t *requires attention* when one of the following holds

(i) $R_t^1\uparrow$, $R_t^2\uparrow$ and $\varphi_e(x_t)\downarrow$

(ii) $R_t^1\downarrow$, $R_t^2\uparrow$ and $\Phi_e(A_{z^i}; x_t)\downarrow$

And R_t *receives attention* when

- If (i) holds then R_t^1 receives attention.
- If (ii) holds then R_t^2 receives attention.

2.3.3 Construction

- *stage* 0. Define $y_0 = 0$ and $z_0^0 = 0.9$.
- *stage* $s+1$.

step A Define

$$z_{s+1}^n = w_s - (n+1)\frac{w_s - y_s}{s+3}$$

for all n with $0 \leq n \leq s+1$.

step B Find the least $t < s+1$ such that R_t requires attention. R_t receives attention (and so, y_{s+1} is defined).

2.3.4 Verification

We start with the following basic

Lemma 11. *At any stage s and for any terms $z_{j_1}^{i_1}, z_{j_2}^{i_2}$ (already defined at s) which do not lie in the black area (i.e. $> y_s$) it is*

$$z_{j_1}^{i_1} > z_{j_2}^{i_2} \iff j_1 < j_2 \vee [j_1 = j_2 \wedge i_1 < i_2]$$

Proof. It follows from the way we define the terms z_t^k in *step A* of the construction by induction on the stages. Indeed, suppose that it holds at (the end of) stage s (it clearly holds at $s = 0$). The terms z_{s+1}^k (for $k \leq s+1$) will be defined less than all the existing terms which do not lie in the black area; so it holds after step A of stage $s+1$. And if there were $z_{j_1}^{i_1} > z_{j_2}^{i_2}$ in the non-black area at the end of $s+1$ (i.e. greater than y_{s+1}) with neither $j_1 < j_2$ nor $[j_1 = j_2 \wedge i_1 < i_2]$, then these two terms should be already defined at the end of step A of the same stage; but we saw that there are no such terms, a contradiction. So the lemma holds after stage $s+1$ and thus the induction step is proved. □

Note that from the above lemma it follows that

$$z_{j_1}^{i_1} = z_{j_2}^{i_2} \Rightarrow (i_1, j_1) = (i_2, j_2)$$

.

Lemma 12. *For every i it is* $\lim y = \lim z^i$.

Proof. By induction, all terms of z^i (for all i) belong in the unit interval. Also, y_s takes values of terms of z^i (for some i) during the construction; so the terms of y also lie in the unit interval and since y is non-decreasing and bounded, it is convergent, say $\lim y = x$. Now fix i. From the construction it follows that

Fact 9.2. *If $x_0 < x$ then there are only finitely many terms of z^i in $(0, x_0)$.*

Claim. *If there is $x_1 > x$ such that infinitely many terms of z^i are in $(x_1, 1)$ then no such term appears in (x, x_1).*

Proof of claim. Suppose otherwise and consider $x < z^i_j < x_1$. Then according to lemma 11, for all $k > i, j$

$$k \notin A_{z^i} \Rightarrow z^i_k \in (x, x_1)$$

This contradicts our assumption. □

By the claim we proved, it suffices to prove that for every $x_1 > x$ there are terms $z^i_j \in (x, x_1)$. Suppose that A_{z^i} is co-infinite. Let $j_1 < j_2 < \dots$ be an enumeration of the infinitely many elements of $\mathbb{N} - A_{z^i}$ (by lemma 11 we have $z^i_{j_1} > z^i_{j_2} > \dots$). We will show that $\lim_n z^i_{j_n} = x$, thus finishing the proof. Note that for all n, $x \leq z^i_{j_n} \leq z^0_{j_n}$, so that it is enough to prove $\lim_n z^0_{j_n} = x$. From the construction it follows that if s_n is the stage where $z^0_{j_n}$ was defined then $s_1 < s_2 < \dots$ (actually $s_n = j_n$). At the beginning of stage s_1 we have the interval $I_{s_1} = (y_{s_1 - 1}, w_{s_1})$ of length ℓ as in the figure below

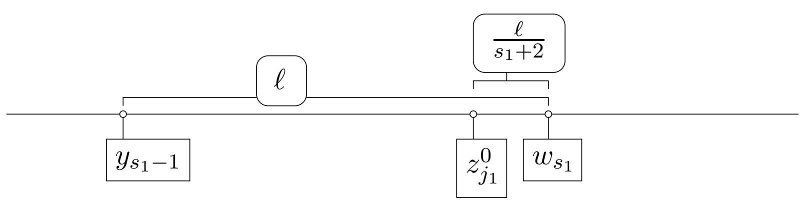

Figure 2.4: The step A of stage s_1 of the construction.

At step A of s_1 we divide I_{s_1} into $s_1 + 2$ equal intervals and set $z^0_{j_1}$ on the first border (and the rest $z^k_{j_1}$, $k \leq j_1$ successively on the other borders according to the construction). Notice that

$$|y_{s_1-1} - z^0_{j_1}| = \frac{s_1 + 1}{s_1 + 2}\ \ell$$

and so

$$\ell = \frac{s_1+2}{s_1+1}\,|y_{s_1-1} - z^0_{j_1}| \tag{2.7}$$

Now during the stages up to s_2, some $z^k_{j_1}$ $(k \le j_1)$ may enter the black area—but not $z^0_{j_1}$, by the choice of s_1. At the beginning of stage s_2 the black area is up to y_{s_2-1} and suppose that $z^k_{j_1}$ is the least term defined at stage s_1 which now is not in the black area. The situation is pictured in the following figure

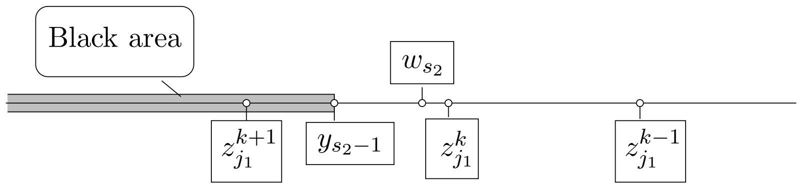

Figure 2.5: The step A of stage s_2 of the construction.

Now $w_{s_2} \le z^k_{j_1}$ and $z^0_{j_2}$ is going to be defined according to step A of the construction in the interval (y_{s_2-1}, w_{s_2}) whose length is obviously $\le \frac{\ell}{s_1+2}$ (the length according to the division done at stage s_1) and so, according to (2.7), $\le \frac{|y_{s_1-1}-z^0_{j_1}|}{s_1+1}$. In the same way one can see that for all n,

$$|y_{s_{n+1}-1} - z^0_{j_{n+1}}| \le \frac{|y_{s_n-1} - z^0_{j_n}|}{s_n+1}$$

so that,

$$|y_{s_{n+1}-1} - z^0_{j_{n+1}}| \le |y_{s_1-1} - z^0_{j_1}| \prod_{i=1}^{n} \frac{1}{s_i+1}$$

But for all n, $y_{s_{n+1}-1} \le x \le z^0_{j_{n+1}}$, hence

$$|z^0_{j_{n+1}} - x| \le |z^0_{j_{n+1}} - x| + |x - y_{s_{n+1}-1}| = |y_{s_{n+1}-1} - z^0_{j_{n+1}}|.$$

Since $\lim_n s_n = \infty$ it is $\lim_n z^0_{j_n} = x$.

Now the case is left where A_{z^i} is co-finite. This means that for almost all j there is s with $y_s > z^i_j$. This, together with fact 9.2 we stated earlier in this proof, gives $\lim_s y_s = \lim z^i$. $\square$

We finish the proof with the following

Lemma 13. *All requirements R_t require attention finitely often and are eventually satisfied.*

Proof. We prove the lemma inductively. Suppose that it holds for all $t < t_0$ and $t_0 = \langle e_0, i_0, j_0 \rangle$. In the following for any t we suppose that $t = \langle e, i, j \rangle$. Choose the last stage s_0 where some R_t, $t < t_0$ received attention. Then R_{t_0} is not going to be initialised after s_0 because, according to the construction, this would mean that some R_t with $t < t_0$ receives attention. Also, at the end of s_0, R_{t_0} was assigned a new witness, say x_{t_0}, with $x_{t_0} > s_0$. So $z^j_{x_t}$ is going to be defined at a stage $s_1 > s_0$ and according to *step A* of the construction it is

$$y_{s_1} < z^{j_0}_{x_{t_0}}$$

and $R^1_{t_0} \uparrow, R^2_{t_0} \uparrow$. Now if $\varphi_e(x_{t_0}) \downarrow$ at some later stage $s_2 > s_1$ (the other case being trivial) then $R^1_{t_0}$ will receive attention since it has the priority. And the relevant action (see section 2.3.2) will be performed, so that

$$y_{s_2} = \max\{z^{i_0}_j \mid j \leq s_2 \wedge z^{i_0}_j > y_{s_2-1} \wedge z^{i_0}_j < z^{j_0}_{x_{t_0}}\}.$$

Note that before $z^{j_0}_{x_{t_0}}$ enters the black area (i.e. $y_s \geq z^j_{x_{t_0}}$, if ever) all subsequent (i.e. after s_2) terms of $z^{i_0}_j$ (that is $z^{i_0}_k$ with $k > s_2$) will appear in $(y_{s_2}, z^{j_0}_{x_{t_0}})$. And since all R_t, $t > t_0$ are assigned new witnesses greater than s_2 at stage s_2, all terms $z^j_{x_t}$ for $t > t_0$ will be in $(y_{s_2}, z^{i_0}_{t_0})$. This means that if some R_t acts (after s_2) before $R^2_{t_0}$ acts, then we will continue to have $y_s < z^{j_0}_{x_{t_0}}$ (i.e. $z^{j_0}_{x_{t_0}}$ outside the black area).

Suppose that it is not the case that $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) = 0$ with use $< \varphi_{e_0}(x_{t_0})$ after s_2. Then one of the following happens:

1. $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) \uparrow$
2. $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) = 1$
3. $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) \downarrow$ with use $\geq \varphi_{e_0}(x_{t_0})$.

In case 1 it is clear that R_{t_0} is not going to require attention from now on and it is trivially satisfied. Otherwise, suppose that the computation *halts* at stage $s_3 > s_2$. In case 2 we note that $z^{j_0}_{x_{t_0}}$ will continue to stay out of the black area for the same reason that it stayed out during the interval of stages between s_2 and s_3 (i.e. because at stage s_2 we initialised all R_t, $t > t_0$ and so the new witnesses will force the respective terms to be defined in $(y_{s_2}, z^{j_0}_{x_{t_0}})$). So for both of the last two cases it suffices to prove the following

Claim. *In the last two cases the computation is going to be preserved in the following stages.*

Proof of claim. By this we mean that no number *below* the use of the oracle $A_{z^{i_0}}$ in the computation is going to enter $A_{z^{i_0}}$ after stage s_3. Indeed, after the convergence at stage s_3, all R_t, $t > t_0$ will be initialised and assigned witnesses greater than s_3. So (according to lemma 11 and the fact that all currently defined terms z_r^k at stage s have $r \leq s$) at any forthcoming redefinition of y (say at stage s_4, caused by some R_t, $t > t_0$) we will still have y_{s_4} less than all terms existing (in the non-black area) at stage s_3. But the use of the computation $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0}) \downarrow$ is less than s_3. So at any forthcoming stage s_4, y_{s_4} will be less than all non-black terms below the use. In other words, no element below the use is going to enter $A_{z^{i_0}}$ (and more generally $\cup_i A_{z^i}$) after s_3 and so the computation will be preserved forever. □

Indeed, now in case 2 the dissagreement will be preserved and in case 3 we will have a computation which is (and will remain) not appropriately bounded.

Now we left the case $\Phi_{e_0}(A_{z^{i_0}}; x_{t_0})[s_3] = 0$ with bound $\varphi_e(x_{t_0})$ which is the one where y is redefined for the sake of $R^2_{t_0}$. In that case, according to the construction, x_{t_0} enters $A_{z^{j_0}}$ (in particular $y_{s_3} = z^{i_0}_{x_{t_0}}$). It suffices to prove the following

Claim. *The computation will not be spoilt by such an action.*

Proof of claim. By the action perfomed at stage s_2, all terms $z_k^{i_0}$ with $z_k^{i_0} < z^{j_0}_{x_{t_0}}$ lying in the non-black area after stage s_2, have $k > s_2$ and thus $k > \varphi_e(x_{t_0})$. So (since $z_k^{i_0} \neq z^{j_0}_{x_{t_0}}$ for all k) all k which go into $A_{z^{i_0}}$ at stage s_3 are greater than the use of the computation and so the last is not spoilt. □

So at the end of stage s_3 we will have the desirable dissagreement and satisfaction of R_{t_0} which will be preserved in the later stages; the last is because the computation will be preserved by the same argument we used in the previous claim. □

2.3.5 Further remarks.

Note that for the sequence $\{z^n\}$ we constructed above, it is

$$A_{z^n} \subseteq A_{z^{n+1}}$$

for all n. Also, by a modification of the definition of z_s^i ($i = 0, \ldots, s$) at stage s (namely we define them such that $z_s^0 < z_s^1 < \cdots < z_s^s$) we get

$$A_{z^n} \supseteq A_{z^{n+1}}$$

for all n. So we have

Corollary 2. *There is a Turing degree which contains an infinite antichain $deg_{wtt}(D_n)$, $n \in \mathbb{N}$ of wtt-degrees with*

$$D_n \subset D_{n+1}$$

for all n. Similar result holds with $D_n \supset D_{n+1}$ in place of $D_n \subset D_{n+1}$.

2.4 A trivial $\mathcal{D}_x^m$.

In the last section we exhibited a c.e. real x whose approximation structure is complicated; namely the distribution of the elements of $\mathcal{D}_x^{\mathrm{wtt}}$ in the Turing degree of x is quite sparse. It is natural to look for the other extreme: are there non-computable reals x such that $\mathcal{D}_x^{\mathrm{r}}$ is trivial (i.e. consisting of a unique element)? Well, if $\mathrm{r} = \mathrm{wtt}$ then the existence of contiguous degrees, i.e. non-trivial Turing degrees which contain a unique wtt degree (a well known result of classical computability theory, see [24]) implies the existence of such reals (due to proposition 4). And this is a concrete example of how the nature of the Turing degree of x is related to the approximation structure of x. We will see however that this relation is not trivial. It is also well known that every non-trivial c.e. Turing degree contains not only infinitely many c.e, m-degrees, but also tt-degrees (again, see [24]). So the structure of m-degrees inside a non-trivial Turing degree is quite rich; and this makes a positive answer to the question whether there is a non-trivial x with $\mathcal{D}_x^{\mathrm{m}}$ trivial interesting. Before giving this answer, we would like to note what makes this fact possible. The reason is that strong reducibilities on the set $\mathcal{S}_x$ give oracle computations with special features. A demonstration of this fact is given in section 2.5 where we show that the positive and btt(1) reducibilities both coincide with the m-reducibility on $\mathcal{S}_x$.

Theorem 10. *There are non-computable c.e. reals x with the property*

$$\left.\begin{array}{l} \lim z = \lim w = x \\ A_z, A_w \ \textit{co-infinite} \end{array}\right\} \Rightarrow A_z \equiv_m A_w$$

The proof is a finite injury priority argument with some special features which we are going to discuss in the following.

2.4.1 Preliminaries

Assume an effective enumeration of the rationals in the unit interval $(0, 1)$, say $\{w_i\}$. Now define

$$w_i^e = w_{\varphi_e(i)}$$

where $\{\varphi_e\}$ is a standard enumeration of all partial computable functions. If $w^e = \{w_i^e\}_{i\in\mathbb{N}}$, then $\{w^e\}_{e\in\mathbb{N}}$ is an enumeration of all partial computable rational sequences in the unit interval. In the following, anything we consider on the real line (e.g. sequences, points etc.) are supposed to be in the unit interval (unless otherwise indicated).

We will construct a c.e. real $x = \lim y$ with y a non-decreasing sequence and a sequence z satisfying the following requirements

$$\begin{aligned}
Q: &\quad \lim z = x \\
P_e: &\quad \varphi_e \neq A_z \\
N_e^r: &\quad w^e \text{ total } \Rightarrow A_{w^e} \leq_m A_z \\
N_e^l: &\quad \left.\begin{array}{l} w^e \text{ total} \\ \lim_s w_s^e = x \\ A_{w^e} \text{ co-infinite} \end{array}\right\} \Rightarrow A_z \leq_m A_{w^e}
\end{aligned}$$

At stage s, y_s, z_s are defined. We need y non-decreasing in order to ensure that x is c.e. and also in order to control the enumeration in the various c.e. sets A_w *for any partial sequence w in the unit interval.* When we say that a number q at a particular stage of the construction is 'in the black area' we mean that it is $y_s \geq q$ (this terminology is motivated by the illustrations, e.g. figure 2.6). Also, $R(e,s) = max\{r(i,s) : i < e\}$; $r(e)$ are the restraints we impose on A_z (with respect to the various negative requirements) and $r(e,s)$ their approximation at stage s ($\lim_s r(e,s) = r(e)$, $\lim_s R(e,s) = R(e)$). We note that in many stages it will be $r(e,s) > s$. But we plan finite injury, so that eventually $r(e) = r(e,s) < s$. In the following, sentences like 'at stage s, $r(e)$ is ...', any parameter considered (like $r(e)$) is supposed to have its current (s-) value. So $r(e,s)$ is sometimes referred to as $r(e)$. It will be clear from the context when $r(e)$ means $\lim_s r(e,s)$. The same applies for other parameters in the construction.

We satisfy P_e by choosing witnesses x_e from the whole pool $\mathbb{N}$ but we also keep priority on the witnesses; this means that at every stage s we have

$$i < j \iff x_i^s < x_j^s. \tag{2.8}$$

Finally we will arrange the construction so that

Fact 10.1. *At stage s we define z_s. If z_k, z_j are not in the black area at stage s then*

$$k < j \iff z_j < z_k.$$

Intuitively this means that there is a tendency to define the terms of z from right to left (i.e. succesively smaller). In contrast, the terms of y are defined from left to right (see e.g. figure 2.6).

2.4.2 Strategies

The strategy for N_e^r

This is how to make A_z of maximum x-m-degree. This strategy is not difficult and it is easily compatible with other requirements: if $w_i^e[s]\downarrow$ and it is not in the *black area* (i.e. less than y_s, see figure 2.6) then in our considerations in defining z_s we take into account w_i^e: we define $z_s < w_i^e$ (for all $i \in \mathbb{N}$ with $w_i^e[s]\downarrow$ which have appeared in the non-black area). The situation is pictured in figure 2.6 where it is seen that we define the current term z_s to be less than all the w^e-terms existing at the time.

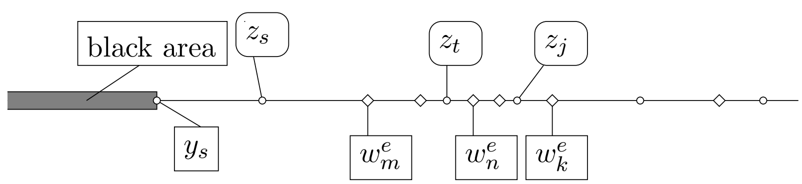

Figure 2.6: The configuration at stage s from the point of view of N_e^r: we define z_s in an interval (y_s, q) which (currently) contains no terms of w^e.

It is clear that we can put all the N_e^r-strategies together. Now the algorithm for $A_{w^e} \leq_m A_z$ (with the hypothesis that w^e is total) is as follows: to answer '$i \in A_{w^e}$?' we wait until a stage s_0 such that $w_i^e[s_0]\downarrow$ and suppose that i is not already in A_{w^e}. We assume the following

Fact 10.2. *At any stage s, $y_s = z_t$ for some $t \leq s$.*

Then, since no z_t with $t > s_0$ will be greater than w_i^e (before the last enters the black area), the only reason why at some stage s it might happen that $y_s \geq w_i^e$ (i.e. $i \in A_{w^e}^s$) is because some term z_t already existing at s_0 and greater or equal to w_i^e enters the black area. So if z_{t_0} is the least such z_t, we have

$$i \in A_{w^e} \iff t_0 \in A_z.$$

The basic strategy for P_e

This is simply wait until $\varphi_e(x_e)\downarrow$ and if it is 0 then put x_e into A_z (i.e. define $y_s \geq z_{x_e}$); otherwise keep x_e out of A_z. Of course, when we choose a witness x_e, it must be $x_e \notin A_z$. One way to do this is: at stage s, choose $x_e^s > s$ *and* set $r(e,s) = x_e^s$ (since we want to continue keeping x_e^s out of A_z until, if ever, it enters A_z under the action of P_e). So, according to (2.8), $r(e,s)$ is increasing in e and non-decreasing in s. Later we will present a more detailed description of the variation of this strategy we are going to use.

The strategy for N_e^l

This is the most important strategy. Viewing the construction from the point of view of N_e^l, the idea is the following: for a particular z_j which is not (yet) in the black area we wait until some w_i^e appears with

$$y_s < w_i^e \leq z_j$$

as in the figure below

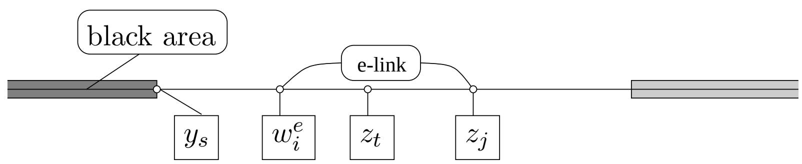

Figure 2.7: Linking z_j for the sake of N_e^l

Then we create a virtual *link* between w_i^e and z_j (called also *e-link* since it is created for the sake of N_e^l), in symbols (w_i^e, z_j), which indicates that (from the point of view of N_e^l) both w_i^e and z_j and any element (e.g. z_t) of the construction appearing between these two should be treated as one and unique point. By this we mean that if at some point we need to put an element of the interval $[w_i^e, z_j]$ into the black area, then *every* element of the interval must enter the black area. Later we are going to involve a different kind of links (the so-called *back links*) in the construction; to avoid confusion, we call the kind of links we just described *front links*; and a *link* is a front or back link. We note that in the following we identify a link (q,p) with the open (unless otherwise indicated) interval of the real line between q and p (so we may say that a real number 'belongs to a link ℓ'); the context will specify the exact meaning of the word. Moreover, we write e.g. $(\ell]$ for the interval $(q,p]$ (where ℓ is the link (q,p)) and if $q > y_s$ at a stage s, we say that the link is outside the black area.

For reference we give the following

Definition 9. *If at a particular stage we have $y_s < w_i^e \leq z_j$ then we say that z_j can be front e-linked.*

By this strategy we can argue $A_z \leq_m A_{w^e}$ roughly as follows: to answer '$j \in A_z$?' we wait until $z_j \downarrow$ and either z_j enters the black area (forever!) or an e-link (w_i^e, z_j) is created. In the last case the link will be present forever and thus $j \in A_z \iff i \in A_{w^e}$. Of course we will have injuries but we plan to have them finitely many, so that we can start the above procedure after a stage beyond which we have no injury of the higher priority requirements.

It will help if we describe the construction intuitively before we state it formally. As we said, at any stage s we have a black area in the unit interval which is the area which the sequence y_s has covered. The black area expands in the course of stages and approaches x. Also, it is universal in its nature i.e. *it does not depend on the way we look at the construction.* This means that with respect to any negative requirement the black area is the same. In the other direction we have

Definition 10. *Suppose that we are at a particular stage s of the construction. We call $r(e)$-white area the (least) upper part of the unit interval which contains all (currently defined) terms of z that are restrained by $r(e)$; that is $[z_{j_0}, 1)$ where z_{j_0} is the least term z_j with $j \leq s$, $j \notin A_z$ and $j \leq r(e)$; by fact 10.1, $j_0 = max\{t : t \leq s \wedge t \notin A_z \wedge t \leq r(e)\}$. Moreover, the e-white area is the union of all $r(i)$-white areas for $i < e$.*

Of course, if $\{t : t \leq s \wedge t \notin A_z \wedge t \leq r(e)\} = \emptyset$, then we define the $r(e)$-white area to be the empty interval (of reals). We notice that the $r(e)$-white area may expand during the stages, although $r(e)$ remains constant (this is because more terms z_j with $j \leq r(e)$ could be defined at later stages). But in this case, after finitely many stages, it will reach the limit

$$[z_{r(e)}, 1)$$

and will remain such unless $r(e)$ changes or its current value enters A_z. Also, we will take care so that $r(e, s) \notin A_z[s]$ at (the beginning of) every stage s. So, it follows that

Fact 10.3. *At every stage s and every e, if $r(e, s) \leq s$ then the $r(e)$-white area is $[z_{r(e,s)}, 1)$.*

Of course the black and the white area are subject to the particular stage s of the construction. The white area is also dependent on the particular 'priority level' from which we view the construction (i.e. the number e). More specifically, our priority list is the following:

$$P_0 > 0\text{-links} > N_0 > P_1 > 1\text{-links} > N_1 > \dots \tag{2.9}$$

where N_e means N_e^l (since we don't have any restraints or positive action for N_e^r). A priority level is a number e and we say that we view the construction from this level when we observe (in the flow of the stages) only the development (actions and restraint modifications) of N_i, P_i for $i \leq e$. In the next section we will see how exactly links are involved in the priority list; in particular, we emphasise that locally we have the following order

$$P_e > e\text{-links} > N_e.$$

The restraints towards the links

Suppose that (q,p) is a link outside the black area at some stage of the construction. Now the negative side of this approach of creating links is that, since all elements in $[q,p]$ are treated as one, this happens also when we define the restraints. In particular, if we install the e-link (q,p) at stage $s+1$ and for some $e_0 \geq e$, p is in the $r(e_0)$-white area and the last does not cover the whole link (see figure 2.8) then we put

$$r(e_0, s+1) = max\{t : q \leq z_t < p\}$$

(assuming that there are such t) and 'initialise' all P_i for $i > e_0$ (see definition 21) in order to make them respect the modified restraint.

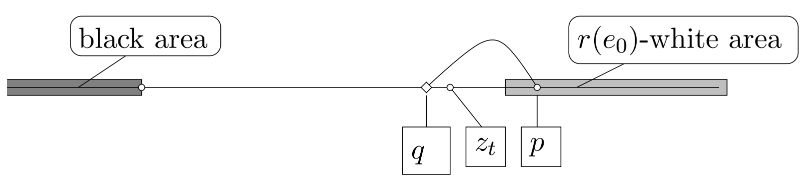

Figure 2.8: Modification of $r(e_0)$ according to the present e-links with $e \leq e_0$.

This means that in figure 2.8 we expand the $r(e_0)$-white area up to the smallest term z_t lying *on* the link. This should happen more generally, when we have a 'chain of links' instead of just one link, as in figure 2.9.

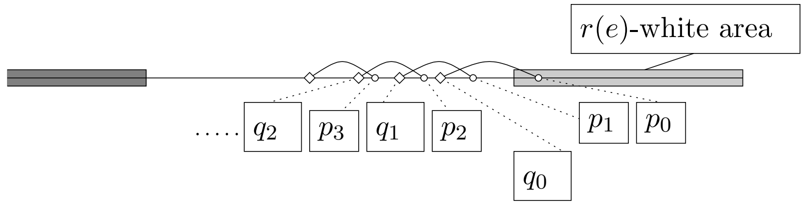

Figure 2.9: A chain of links.

In order to be more precise, we give the following definitions.

Definition 11. *A (finite) chain is a finite sequence of links* $(q_0, p_0), \ldots, (q_n, p_n)$ *existing at a given stage* s*, which are currently not in the black area (i.e.* $\forall i \leq n,\ q_i, p_i > y_s$*) and such that* $q_n < q_{n-1} \leq p_n < q_{n-2} \leq p_{n-1} < q_{n-3} \leq \cdots \leq p_2 < q_0 \leq p_1 < p_0$.

A chain looks as in the following figure

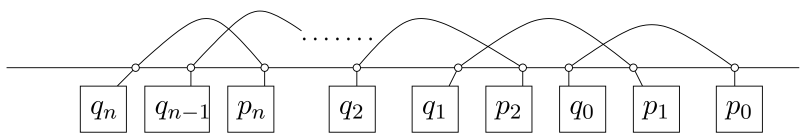

Figure 2.10: The chain of definition 11.

Note that in the above definition, not all links must be front links; in other words links can be 'back links' (a kind of link we are going to define later). Now the following definition makes precise how restraints 'travel' through links or chains of links.

Definition 12. *Suppose that a link* (q, p) *exists at a certain stage* s *outside the black area and for some* e_0, t *it is* $r(e_0) < t \leq s$, $z_t \in [q, p)$ *and* $z_{r(e_0)} \leq p$ *(see figure 2.8). Then we say that the restraint* $r(e_0)$ *can travel through the link; when we say that we travel* $r(e_0)$ *through that link we mean that we put*

$$r(e_0) = max\{t : q \leq z_t < p\} \tag{2.10}$$

Moreover $r(e)$ *can travel through a set of links* $\mathcal{S}$ *which lie outside the black area, if it can travel through at least one of the links in* $\mathcal{S}$*. When we say that we travel* $r(e)$ *through the set of links* $\mathcal{S}$ *we mean that we successively travel* $r(e)$ *through a sequence of links* $\ell_0, \ell_1, \ldots, \ell_n$ *in* $\mathcal{S}$ *such that*

1. $r(e)$ *can travel through* ℓ_0 *and for all* $i < n$*, after it has travelled* ℓ_i *it can travel* ℓ_{i+1}.

2. *After the last trip through* ℓ_n*,* $r(e)$ *cannot travel through any of the links in* $\mathcal{S}$.

This sequence of links is called a path.

Lemma 14. *If $r(e)$ travels through a set of links $\mathcal{S}$ (at a stage s) then the path used (say $\ell_0, \dots, \ell_n$) forms a maximal chain in $\mathcal{S}$ (i.e. $\forall \ell \in \mathcal{S}$, the sequence $\ell_0, \dots, \ell_n, \ell$ is not a chain). Moreover, if it travels through different paths then the final value of $r(e)$ will be the same.*

Proof. Suppose that the path used (say $\ell_0, \dots, \ell_n$) is not a chain. Then there is a least i_0 such that $\ell_0, \dots, \ell_{i_0}$ is a chain but $\ell_0, \dots, \ell_{i_0+1}$ is not. Now by definitions 12 and 11 it follows that after travelling through ℓ_{i_0}, $r(e)$ could not travel through ℓ_{i_0+1}, a contradiction.

Now if this path is not maximal as a chain, there would be a link $\ell_{n+1} \in \mathcal{S}$ such that $\ell_0, \dots, \ell_{n+1}$ is a chain. But this would imply that $r(e)$, after travelling through ℓ_{i_n}, can travel through another link of $\mathcal{S}$ (links in a chain are distinct by definition) which contradicts definition 12.

Now suppose that when $r(e)$ travels through the paths $\ell_0, \dots, \ell_n$ and $h_0, \dots, h_m$ it yields different values (e.g. the value with the first is less than the one with the second trip). The value of $r(e)$ after the first trip is

$$[r(e)]_1 = max\{t : z_t \in \ell_n\}$$

(where ℓ_n is considered as a closed interval here). Find the maximum link of the second path (in the ordering $h_0, \dots, h_m$) say (q, p), such that $p \geq z_{[r(e)]_1}$. We claim that there is $z_{t_0} \in [q, p)$ with $z_{t_0} < z_{[r(e)]_1}$. Indeed, if not then $r(e)$ would have not reason to travel the successor link of (q, p) (according to the second trip). But since there is a link $(q, p) \in \mathcal{S}$ with the above properties, the first path is not a maximal chain! This contradicts the first part of the above proposition. So we must have $[r(e)]_1 = [r(e)]_2$. □

By the above proposition when we state the construction we can say e.g. 'travel $r(e)$ through the existing links' (and the new value of $r(e)$ will be uniquely determined).

The travelling of restraints we described in this section *will prevent a link from allowing a positive action to injure a higher priority restraint.* Indeed, suppose that P_{e_1} with $e_1 > e$ would like i in A_{w^e} but N_{e_0} with $e_0 < e_1$ would like j out of A_z (and the link (q, p) of figure 2.11 exists).

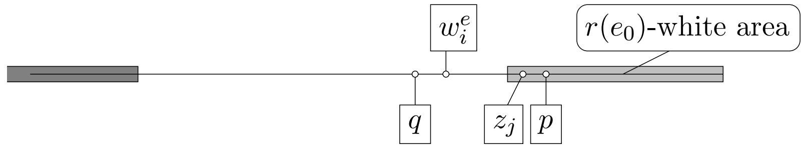

Figure 2.11: The priority amongst P_e, links and N_e.

According to the priority we set out in (2.9) we have:

Principle 1. *The e-links are visible (taken into account) only by the requirements* P_t, N_k^l, *with* $t > e$ *and* $k \geq e$. *In particular,* $r(e_0)$ *can only travel e-links with* $e \leq e_0$. *Similarly, when y is redefined for the sake of* P_t *(see definition 19), it can only 'travel' (see definition 19) e-links with* $e < t$.

The link in the above case creates conflict amongst two requirements which otherwise (i.e. if links where not involved in the construction) would not exist. The link should definitely be taken into account when P_{e_1} acts, since it is visible from that requirement (see principle 1). But of course, for priority reasons, the negative requirement N_{e_0} should also be taken into account by P_{e_1}. This means that *in this case we decide not to act*, thus respecting the priority of the requirements as usual: that is why we define (2.10); by this modification of the restraints, P_{e_1} will be assigned new witness and prevented from injuring higher priority requirements.

Now if we had $e_1 \leq e$ in the situation described above, then the e-link is not visible by P_{e_1} and thus the last need not take it into account at all. In this case i will enter A_{w^e} as P_{e_1} wants it but j will stay out of A_z: so the link (q, p) will be *cancelled*.

Definition 13. *Suppose that during a particular stage s there is a link* (q, p) *such that q is in the black area but p is not. Then we say that the link is half-black.*

We plan to cancel any *half-black* links at the very stage they appear:

Principle 2. *If at some stage there is a link* (q, p) *with* $q \leq z_t \leq p$ *and* z_t *enters the black area but p stays out of it, then the link is cancelled and never considered in the following stages.*

Note that all the above apply also for *back-links*, a kind of links we are going to define later.

Finally, if it was $e_1 > e$ and $e_1 \leq e_0$ then N_{e_0} should be injured by P_{e_1} (as in a typical priority argument) and the link will be, as we say, *travelled* (by y, see definition 19) (i.e. all t with $z_t \in [q, p]$ go into A_z).

Problems with the restraints.

Now one can see that creating arbitrarily many links may cause a single restraint to go to infinity. A typical such situation is when the chain in figure 2.9 is infinite.

Definition 14. *An infinite chain is an infinite sequence of links* (q_0, p_0), $(q_1, p_1), \dots$ *created in the course of the construction, such that*

- *the links never go into the black area.*
- *for all* k, $p_0 > p_{k+1} \geq q_k > p_{k+2}$.
- *none of the links is cancelled during the construction.*

Here the vicious situation starts from a term p_0 (see figure 2.9) which happens to be *in* an $r(e)$-white area. Within a link of this term (q_0, p_0) there is a term of z which causes the $r(e)$-white area (which represents the e-th restraint for A_z) to expand up to p_1. Now the same happens with another term of z and a p_2 in an e_2-link (q_1, p_1) and by creating links in such a fashion indefinitely during the construction, we make the $r(e)$-white area moving towards x without becoming eventually constant. This behaviour of the restraint $r(e, s)$ which now goes to infinity in the course of stages s, may prevent a positive requirement from fulfilling its purpose.

Bounding the restraints.

To prevent the situation $\lim_s r(e, s) = \infty$ (arising from the existence of *infinite chains* travelled by a single restraint) we will be more careful when we are installing links; namely, we will do that only when we *really* need it.

Links only for terms outside the white area We remark that

Principle 3. *We need to e-link* z_j *only when* $j \notin A_z[s]$ *and* $j > R(e, s)$.

And this is because in the verification of $A_z \leq_m A_{w^e}$ (described above) we can assume we know $R(e)$ (i.e. the final value of $R(e, s)$) *a priori* so that when we are asked about '$j \in A_z$?' with $j \leq R(e)$ we will be able to answer directly (j is in A_z only if it is there by the time $R(e)$ takes its final value). In other words, even if we have the above restriction in installing links, we can still be sure that when we run the stages (after a stage where $R(e)$ becomes constant) *we either find* j *enumerated in* A_z *or* $j \leq R(e)$ (in which case if it is out, it will stay out forever) *or we find* z_j *e-linked, in which case* *the link will stay there forever, thus giving us the answer depending on a unique term of* w^e.

What we want is to 'e-settle' every term of z, for every e such that the hypotheses of N_e^l are satisfied; when we say that z_j is e-settled at some stage, we mean that one of the following cases is realized (e-*linked* means front or back e-linked; back links are defined in the next section):

- z_j has entered the black area.
- z_j has entered the e-white area.
- z_j is e-linked.

Back links However we need one more trick on the production of links in order to avoid restraints travelling indefinitely. The idea is that instead of creating a front link for z_j, we can alternatively create a so-called 'back link' for it; that is simply link z_j with a term w_i^e with $w_i^e \geq z_j$. And if we prefer back links rather than front ones in a chain travelled by $r(e)$, the restraint cannot travel indefinitely; this is because a restraint cannot travel links which are situated in its own (white) area. In particular, we will prefer to back e-link z_j when we think that creating a front e-link instead, will force some $r(t)$ for $t > e$ to travel it; that is when z_j is in the $r(t)$-white area for some $t > e$. Also, because we do not want to make higher priority restraints to travel, we will require that w_i^e is not in the t-white area.

Definition 15. *We say that* z_j *can be back* e*-linked at stage* s *when it is in the* $r(t)$*-white area for some* $t > e$ *and there are terms* w_i^e *(*$w_i^e \downarrow$ *by stage* s*),* z_v *(*$v \leq s$*), not lying in the* t*-white area, with* $z_j < w_i^e \leq z_v$.

The reason why we involved z_v in the above definition is not obvious; we did this in order to realize fact 10.2 which was assumed when we sketched why the strategy for N_e^r works, in section 2.4.2. Namely, if y ever travels that back link (say at stage s_1), we will arrange that $y_{s_1} = z_v$ instead of merely $y_{s_1} = w_i^e$ (see definitions 19, 20). Because we do not want to injure requirements $r(m)$ for $m < t$ by such an action, we require z_v to be outside of the t-white area. So we treat the elements in $[z_j, z_v]$ as one, and thus we rather say that we link z_j with z_v (instead of w_i^e) and the link is written as (z_j, z_v).

We are now able to argue that while enumerating the links in order to answer '$j \in A_z$?', our search will *halt* giving an answer based on an m-query to A_{w^e}. In fact, one of the following will happen:

(1) z_j enters the black area

(2) z_j is in the e-white area

(3) z_j becomes front e-linked

(4) z_j becomes back e-linked

In other words we will witness that z_j has been *e-settled* according to the following

Definition 16. *We say that z_j is e-settled at stage s when one of the following cases holds:*

- *z_j has entered the black area ($z_j \leq y_s$)*
- *z_j has entered the e-white area ($z_j > y_s \wedge j \leq R(e, s)$)*
- *z_j is front e-linked*
- *z_j is back e-linked*

We say that it is ready to be e-settled when it can be front or back e-linked at the current stage.

Definition 17. *We say that z_j is settled at stage s when it is e-settled for every e. We say that it is ready to be settled when it is ready to be e-settled for some e.*

Note that we will have $r(e, s)$ increasing in e; so, for every stage s and term z_j there is a unique t such that z_j is in the $r(t)$-white area but not in the t-white area.

Definition 18. *Suppose that z_j is ready to be e-settled and it belongs to the $r(t_0)$-white area but not in the t_0-white area. We say that we e-settle z_j when*

- *if it can be back e-linked, we back e-link it with the least z_v with $z_v \geq w_i^e > z_j$ (for some w_i^e), not lying in the t-white area.*
- *Otherwise we e-link it with the largest w_i^e (with $w_i^e \leq z_j$ and not lying in the black area) available.*

We note that we will not succeed in settling all terms z_j. What is needed is to e-settle all z_j just for the e such that the hypotheses of N_e^l hold. If we succeed this then we can give an m-oracle procedure to answer the question '$j \in A_z$?' with the help of A_{w^e} as it was described above.

Why $\lim_s r(e,s) < \infty$ We make the following remarks

Remark 1. *Consider a finite chain as in the following figure*

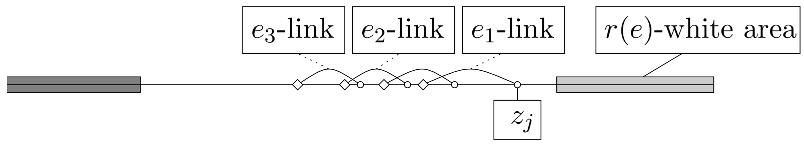

Figure 2.12: Travelling of $r(e)$ in a single stage.

in a stage where z_{j_0} is out of the $r(e)$-white area and $e_i \leq e$. We will arrange the construction so that the following holds: if at a later stage s_1, z_j enters the $r(e)$-white area (and none of the links in the chain is cancelled by that stage), at this very stage the $r(e)$-white area will be forced to cover all terms of z which lie on a link of the chain. So, when a link which belongs in such a finite chain is travelled by a restraint $r(e)$, then all the links in the chain are travelled at the same time.

Remark 2. *At any stage s only finitely many links are travelled by a restraint $r(e)$. This is because finitely many links exist at s.*

Now we explain why $\lim_s r(e,s) < \infty$; if this is not true, there is a least restraint $r(e)$ which travels through an infinite chain of links during the construction. There are infinitely many stages in which $r(e)$ travels front links and at a single such stage it will travel a finite chain of links (according to the above remarks). We can assume that this infinite chain is not in the e-white area (since e is the least with $\lim_s r(e,s) = \infty$). The situation is pictured in the following figure

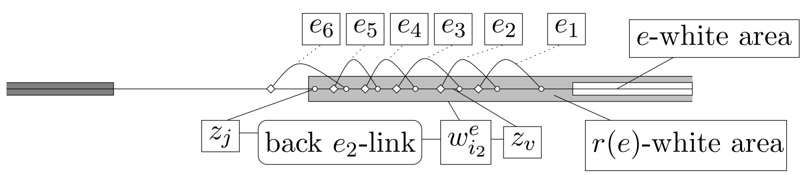

Figure 2.13: The role of back links.

By principle 3 we have that all the links of the infinite chain we consider are i-links with $i \leq e$. This means that after some stage there will be terms $w_t^{e_k}$ in the $r(e)$-white area but not in the e-white area, for all k (see figure 2.13). And since we give priority to *back* linking for terms z_j like the one pictured above, we will stop issuing front e_k-links for such terms (the requests for e_k-linking will be satisfied with back e_k-linking); and $r(e)$ cannot travel any back e_k-link created in its own white area (i.e. the $r(e)$-

white area). This is a rough explanation why this chain cannot expand forever; a more detailed proof of this fact is given in the verification.

More about P_e.

For the satisfaction of P_e we act as usual: P_e can be in the state of 'satisfied' at a particular stage s ($P_e[s] \downarrow$) or in that of 'unsatisfied' ($P_e[s] \uparrow$). We say that it *requires attention at* s when $P_e[s] \uparrow$ and

$$\varphi_e(x_e)[s] \downarrow = 0 \tag{2.11}$$

As usual, the satisfaction of P_e may be achieved by putting x_e into A_z. This in turn is achieved by the definition of y_s, i.e. by expanding the black area. Because of the presence of links, this may take several steps which we view as substages of the stage s. At each of these intermediate steps we have an approximation y_s^n to y_s and after finitely many steps, this process of generating $y_s^0, y_s^1, \dots$ will come to an end, giving the value of y_s. For reference we give the following

Definition 19. *We say that y_s^n can travel through a (back or front) link (q,p) at a given stage s when $q \leq y_s^n < p$. And y_s^n travels through that link when we have $y_s^{n+1} = p$.*

Definition 20. *Lets write $\mathcal{L}_e^s$ for the set of the e-links which exist at stage s (after the end of step B of the construction, see section 2.4.3). When we say that P_e receives attention at stage $s+1$, we mean that the following action is performed*

1. *Define*

$$\begin{aligned} y_{s+1}^0 &= z_{x_e^s} \\ y_{s+1}^{n+1} &= \max\{\sup \ell \mid \exists m[m < e \wedge \ell \in \mathcal{L}_m^{s+1}] \wedge y_{s+1}^n \in [\ell)\} \end{aligned}$$

where in the expression $\sup \ell$, *ℓ is considered as an interval. After some n, $y_{s+1}^n \uparrow$ (since the set to which the* $\max$ *applies will be empty). Now if $i_0 = \mu t[y_{s+1}^{t+1} \uparrow]$ define*

$$y_{s+1} = y_{s+1}^{i_0}.$$

This is a formal way to say that, we first put $y_{s+1}^0 = z_{x_e^s}$ (thus enumerating x_e^s into A_z) and then we travel successively $y_{s+1}^0, y_{s+1}^1, \dots$ through any m-links with $m < e$

that can travel them [3], *until we reach some* $y^{i_0}_{s+1}$ *which cannot travel through any existing link of this kind: this is the value of* y_{s+1}.

2. *Put* $P_e[s+1]\downarrow$.

Definition 21. *To initialise the requirements* P_i *for* $i > e$ $(e \geq -1)$ *at stage* s *means to set*

- $x^s_{e+1} = \mu t[t > s+1 \wedge t > e+1]$ *and for* $i > e$, $x^s_{i+1} = \mu t[t > x^s_i]$.
- $r(i,s) = x^s_i$ *for* $i > e$.
- $P_i[s]\uparrow$.

2.4.3 Construction

Stage 0 Initialise all P_e, $e > -1$ and set $y_0 = 0, z_0 = 0.9$.

Stage s+1

step A Define z_{s+1} to be in the middle of (y_s, w) where

$$w = min\{w^e_i, z_t, 1 : t \leq s \wedge e, i \in \mathbb{N} \wedge w^e_i[s+1]\downarrow \wedge w^e_i, z_t > y_s\} \tag{2.12}$$

step B B_1 Find the least j such that z_j is not settled and is ready to be settled. Then, find the least e, say e_0, such that z_j is not e-settled and is ready to be e-settled; e_0-settle z_j.

B_2 Now we travel the least restraint $r(e)$ that can travel through the existing i-links for $i \leq e$. Also, initialise all P_i, $i > e$.

step C Find the least P_e $(e \leq s)$ which requires attention and

C_1 P_e receives attention : y_{s+1} is defined.

C_2 Initialise all P_i for $i > e$ and put $r(e, s+1) = s+2$.

C_3 Cancel any half-black links.

Note the redefinition of $r(e)$ at step C_2; this is done in order to realize $r(e,s) \notin A_z[s]$ which we promised just before stating fact 10.3.

[3] for a y^n_{s+1} there may be more than one ways (links) to travel it. In the formal definition we choose the link which will make y^{n+1}_{s+1} maximum; but it is easy to see that any other choice would lead to the same definition of y_{s+1}.

2.4.4 Verification

Lemma 15. *If z_k, z_j are not in the black area at a particular stage s then*

$$k < j \iff z_j < z_k.$$

Proof. It follows from the way we define z_{s+1} in *step A* of the construction by induction on the stages. Indeed, suppose that it holds at (the end of) stage s (it clearly holds at $s = 0$). The term z_{s+1} will be defined less than all the existing terms of z which do not lie in the black area; so it holds after step A of stage $s+1$. And if there were $z_k < z_j$ with $k < j$ in the non-black area at the end of $s+1$ (i.e. greater than y_{s+1}), then these two terms should be already defined at the end of step A of the same stage; but we saw that there are no such terms, a contradiction. So the lemma holds after stage $s+1$ and thus the induction step is proved. □

Lemma 16. $\lim y = \lim z$

Proof. By induction, all terms of z belong in the unit interval. Also, y_s takes values of terms of z during the construction; so the terms of y also lie in the unit interval and since y is non-decreasing and bounded, it is convergent, say $\lim y = x$. Now from the construction it follows that

Fact 10.4. *If $x_0 < x$ then there are only finitely many terms of z in $(0, x_0)$.*

Claim. *If there is $x_1 > x$ such that infinitely many terms of z are in $(x_1, 1)$ then no term of z appears in (x, x_1).*

Proof of claim. Suppose otherwise and consider $x < z_{j_0} < x_1$. Then according to lemma 15, for all $j > j_0$, $j \notin A_z \Rightarrow z_j \in (x, x_1)$. This contradicts our assumption. □

By the claim we proved, it suffices to prove that for every $x_1 > x$ there are terms $z_j \in (x, x_1)$. Suppose that A_z is co-infinite. Let $j_1 < j_2 < \dots$ be an enumeration of the infinitely many elements of $\mathbb{N} - A_z$ (by lemma 15 we have $z_{j_1} > z_{j_2} > \dots$). We will show $\lim_s z_{j_s} = x$, thus finishing the proof. From the construction it follows that if s_n is the stage where z_{j_n} was defined then $s_1 < s_2 < \dots$ (actually $s_n = j_n$) and

$$z_{j_n} = y_{s_n-1} + \frac{\lambda_{s_n} - y_{s_n-1}}{2}$$

where λ_{s_n} is the minimum of 1 and all z_j, w_i^e which have appeared by the end of stage $s_n - 1$ and are not yet in the black area (see (2.12) in the construction). Clearly we have $\lim y = x \geq y_n$ and $z_{j_n} \geq \lambda_{s_{n+1}}$ for all n. So, if

$$\begin{aligned} a_1 &= z_{j_1} \\ a_{n+1} &= x + \frac{a_n - x}{2} \end{aligned}$$

is a recursively defined sequence, then for all n

$$\begin{aligned} a_n &\geq z_{j_n} \\ a_n &> a_{n+1} \end{aligned}$$

So it is enough to prove that $\lim_n a_n = x$. But this is not difficult to do (we omit the proof).

Now the case is left where A_z is co-finite. This means that for almost all j there is s with $y_s > z_j$. This, together with fact 10.4 we stated earlier in this proof, gives $\lim y = \lim z$. □

Lemma 17. *N_e^r are satisfied.*

Proof. Suppose that the hypotheses of N_e^r hold. It is enough to prove $A_{w^e} \leq_m A_z$. To answer '$i \in A_{w^e}$?' wait until a stage s_0 with $w_i^e[s_0] \downarrow$ and suppose that $w_i^e > y_{s_0}$ (otherwise we answer positively). By the construction, every time y_s is redefined, it is $y_s = z_k$ for some $k \leq s$. Also,

$$s > s_0 \Rightarrow z_s < w_i^e$$

as long as w_i^e stays out of the black area. This means that, if $z_{j_0} = min\{z_t : t \leq s_0 \wedge z_t \geq w_i^e\}$ then

$$i \in A_{w^e} \iff j_0 \in A_z$$

Of course, if $\{z_t : t \leq s_0 \wedge z_t \geq w_i^e\} = \emptyset$ then $i \notin A_{w^e}$. □

Lemma 18. *If an e-link is cancelled at stage s then at the same stage some P_i with $i \leq e$ receives attention.*

Proof. Suppose that the link (q, p) is cancelled at s. From the construction it follows that some P_i receives attention at s and that (q, p) becomes half-black during part C_1 of the stage s. So $q \leq y_s < p$. But $y_s = y_s^n$ for some n such that y_s^n cannot travel through any of the existing m-links with $m < i$. Now if it was $i > e$, the link (q, p) would be visible from P_i and according to definition 20, y_s^n would travel it. But this would mean that this link is not half-black, a contradiction. □

Lemma 19. *A term z_j is settled at stage s ($s \geq j$) iff it is e-settled for every $e < j$.*

Proof. By definition 17 if z_j is settled, then it is e-settled for all $e < j$. So it remains to prove the converse. By induction on the stages of the construction it follows that $r(e, s)$ is non-decreasing in s for every e. Also, by definition 21 and step 0 of the construction, we have $\forall e$, $r(e, 0) > e$. So,

$$\forall e \forall s[r(e, s) > e \wedge R(e, s) \geq e]$$

This means that for all $e \geq j$, z_j is in the e-white area (if it is not in the black one) and thus it is automatically e-settled due to definition 16. So if it is e-settled for every $e < j$, then it is e-settled for all e, i.e. settled. □

Lemma 20. *A requirement P_e never injures a restraint with higher priority; i.e. if P_e acts at s, no number $m \leq R(e, s)$ enters A_z at that stage.*

Proof. Suppose that P_{e_0} acts at s_0 (under step C_1 of the construction) and $m \leq R(e_0, s_0)$ *enters* A_z.

Claim. *$r(e, s)$ is increasing in e at every s.*

Proof of claim. This holds at stage 0 by definition 21. Suppose that it holds at (some step of) stage s. At the next step, either all $r(e)$ remain the same or some $r(e)$ increases under step B_2 and all $P_i, i > e$ are initialised, or we just initialise all $P_i, i > e$ under step C_2. In any case the claim continues to hold due to definition 21. □

So we have $R(e, s) = r(e-1, s)$ and in particular $R(e_0, s_0) = r(e_0 - 1, s_0)$ and

$$m \leq r(e_0 - 1, s_0). \tag{2.13}$$

On the other hand we have

Claim. $\forall e, s,\ r(e, s) \geq x_e^s$

Proof of claim. At stage 0 it holds. If it holds at a particular step of a stage, then in the next step either $r(e)$ increases via step B_2 or all $P_i, i > t$ for some $t < e$ are initialised via step B_2 or C_2 (or both x_e, $r(e)$ remain the same). In any case, the claim continues to hold due to definition 21. □

Note that as long as $P_e \uparrow$, if $r(e, s) > x_e^s$ at some stage, then $r(e)$ has travelled some links; so, by definition 12, it is $r(e, s) \leq s$. This, along with the last claim gives

$$x_e^s \leq s \Rightarrow r(e, s) \leq s$$

for all e, s. Now since P_{e_0} acted at s_0, it must be $x_{e_0}^{s_0} \leq s_0$. So $r(e_0, s_0) \leq s_0$ and in particular $r(e_0 - 1, s_0) \leq s_0$; and because of (2.13) and lemma 15, $z_{r(e_0-1)} \leq z_m$. This means that by the action of P_{e_0}, $r(e_0 - 1)$ was also enumerated in A_z. Now by an induction on the stages (and substages) of the construction one can prove that

$$\forall e, s,\ x_e^s > R(e, s).$$

So we have $x_{e_0}^{s_0} > R(e_0, s_0)$ and according to the above,

$$z_{x_{e_0}} < z_{r(e_0-1)} \leq z_m \tag{2.14}$$

at s_0.

Claim. *At step C_1 of the construction a chain $\ell_1, \dots \ell_n$ of e-links with $e < e_0$ was travelled by y.*

Proof of claim. Suppose otherwise; then by construction, $y_{s_0} = z_{x_{e_0}}$. So, by 2.14, m stays out of A_z at s_0; a contradiction. □

For the chain of the above claim, it is $z_{x_{e_0}} \in [\ell_n)$ and $z_{r(e_0-1)} \in (\ell_i]$ for some $i \leq n$ (the last because m goes into A_z and (2.14)). Suppose that i is the maximum such that $z_{r(e_0-1)} \in (\ell_i]$. We have

Claim. *At step B_2 of s_0, $r(e_0 - 1)$ would travel ℓ_i.*

Proof of claim. Suppose that $i = n$. Then it follows from (2.14) and the fact $z_{x_{e_0}}, z_{r(e_0-1)} \in [\ell_n]$ that $r(e_0 - 1)$ will travel ℓ_n. Now suppose that $i < n$. Then, if $\ell_{i+1} = (q, p)$, it is $p \in [\ell_i)$ and $p < z_{r(e_0-1)}$ (otherwise i would not be maximum such that $z_{r(e_0-1)} \in (\ell_i]$). And p is a term of z, so that $r(e_0 - 1)$ will travel ℓ_i. □

As a result, P_{e_0} would be initialised. This is a contradiction since the new witness will not satisfy $z_{x_{e_0}} \in [\ell_n)$. □

Lemma 21. *For all e, $\lim_s r(e, s) < \infty$ and P_e receives attention finitely often; also, if φ_e is total then P_e is satisfied.*

Proof. We prove this by induction on the priority list $P_0 > N_0 > P_1 > N_1 > \dots$. That the lemma is true for P_0 it is easy to see. Now suppose that the lemma is true for all $i < i_0$; and choose the least stage s_0 after which no P_i with $i < i_0$ receives attention and no $r(i)$ changes its value. At s_0, P_{i_0} has a current witness, say x_{i_0}. P_{i_0} is not going to be initialised under step B_2 (after s_0) because otherwise some $r(e)$ with $e < i_0$ would travel and change its value. Also, it will not be initialised under C_2, because otherwise some P_i with $i < i_0$ receives attention. Now if P_{i_0} receives attention while it has x_{i_0} as a witness (the *last* witness), then it is $\varphi_{i_0}(x_{i_0}) \downarrow = 0$ and x_{i_0} will enter A_z (since it has the priority amongst the positive requirements) so P_{i_0} is satisfied for ever ($P_{i_0} \downarrow$ and it never requires attention again). Otherwise, suppose that $\varphi_{i_0}(x_{i_0}) \downarrow = 1$. Then x_{i_0} is restrained from A_z with priority i_0, *and it is going to stay restrained.* This is because the restraints $r(e, s)$ are non-decreasing in s; and since no P_i with $i < i_0$ is going to receive attention, by lemma 20 we have that x_{i_0} will stay out and so P_{i_0} is satisfied. The case when $\varphi_{i_0}(x_{i_0}) \uparrow$ is trivial.

To complete the proof it is enough to prove $\lim_s r(i_0, s) < \infty$. Choose a stage s_1 after which no P_i with $i \leq i_0$ receives attention and all $r(i)$ for $i < i_0$ have reached their limit. For a contradiction, suppose that $\lim_s r(i_0, s) = \infty$ (it cannot be otherwise since $r(i_0, s)$ is non-decreasing in s). We claim that step B_2 is performed infinitely many times for the sake of N_{i_0}; indeed, if not, then because of our assumptions about s_1, $r(i_0)$ would not change afterwards (since step C_1 would not be performed for P_i with $i \leq i_0$), a contradiction. It follows that there is an infinite chain of links (see definition 14) such that $r(i_0)$ travels through it at infinitely many stages. By construction we have that all links occurring in the chain are e-links for $e \leq i_0$ which never enter the black area. Indeed, when they were travelled by $r(i_0)$ they were outside the black area; and later they were *in* the $r(i_0)$-white area and thus protected with priority i_0. So, since no P_i with $i \leq i_0$ is going to act, they will never get injured (i.e. cancelled) or enter the black area.

So there is a finite set B which consists of all e such that infinitely many e-links occur in the infinite chain. Choose a stage $s_2 > s_1$ beyond which $r(i_0)$ travels only e-links with $e \in B$ and it has already travelled e-links for every $e \in B$ since stage s_1. At some $s_3 > s_2$, $r(i_0)$ is going to travel again a finite chain and at the last link of this chain it will stop because of the lack of a suitable link (only finitely many links exist at a particular stage). Note the following

Fact 10.5. *It is $r(i_0) \notin A_z$; so the $r(i_0)$-white area is $[z_{r(i_0)}, 1)$.*

In case P_{i_0} acted after s_0, this follows from the second action in step C_2 of the construction; also because, since $r(e)$ has started travelling and is not going to be initialised in later stages, it is $r(e, s) \leq s$ for all $s > s_2$. By the next (i.e. after s_3) stage where $r(i_0)$ is travelling again, a suitable link will have appeared; this is an e-link ℓ with $z_{r(i_0)} \in (\ell]$.

Claim. *This link cannot be back e-link.*

Proof of claim. Suppose otherwise. When $r(i_0)$ stopped travelling at stage s_3, ℓ did not exist. And such back link cannot be created in later stages by definition 15. Indeed, ℓ would have appeared for the back e-linking of some z_j which is situated in the $r(t)$-white area for some $t > i_0$. But then, by definition 15, $(\ell]$ does not intersect the t-white area and so the $r(i_0)$-white area (which is contained in the former). So $z_{r(i_0)} \notin (\ell]$, a contradiction. □

So ℓ is a *front* link. Now we claim that such a link should not appear, arriving thus to a contradiction; indeed, when some $z_j \geq z_{r(i_0)}$ asked for e-linking, according to the construction we first look whether we can create a back link (w.l.o.g. suppose that z_j is not in the i_0-white area). If this is not possible, i.e. there is no w_i^e, z_v with $z_j < w_i^e \leq z_v < z_t$ (where $[z_t, 1)$ is the i_0-white area) then by the choice of s_2 there must be a term w_i^e with $z_{r(i_0)} \leq w_i^e \leq z_j$ (otherwise $r(i_0)$ would not have travelled an e-link after stage s_1, a contradiction). So, since we front e-link z_j with *the least* w_i^e available, such a link should not appear, a contradiction. □

Lemma 22. *For every $j \in \mathbb{N}$ and e such that $x = \lim w^e$ and A_{w^e} is co-infinite, there is a stage s_0 in which z_j is settled and any links (q, z_j), (z_j, p) ($e \in \mathbb{N}$) are never cancelled (so it remains settled in later stages).*

Proof. Suppose not. Then there is a least j_0 for which the lemma does not hold. Choose a stage s_0 after which, for all $i \leq j_0$, P_i does not receive attention, $r(i)$ has reached its limit, and every z_j with $j < j_0$ which is to be settled, has already been settled. After this stage no $j < j_0$ will receive attention under step B_1 of the construction. Given j_0, choose e_0 to be the least e such that the lemma does not hold for j_0, e_0. Also, choose a stage $s_1 > s_0$ such that for every $i < e_0$, if z_{j_0} is to be i-settled, it is already so. Then, after that stage, z_{j_0} will not receive attention for i-linking with $i < e_0$.

Now by the hypothesis that z_{j_0} does not satisfy the lemma, we have that $j \notin A_z$ and that there are arbitrarily close terms of w^{e_0} to x from the right. This means that at some stage after s_1, z_{j_0} will be ready to be e_0-settled (if it has not done so far) and it will be e_0-settled immediately since it has the priority; and the link by which it is settled will never be cancelled (by lemma 18 and the assumption that no P_i with $i \leq j_0$ receives attention). This is a contradiction. □

Lemma 23. *N_e^l are satisfied.*

Proof. Suppose that $\lim w^e = x$ and A_{w^e} co-infinite. Choose a stage s_0 after which no P_t with $t \leq e$ acts. To answer '$j \in A_z$?' look for a stage $s > s_0$ such that

(1) it has entered the black area; or

(2) it has entered the e-white area; or

(3) z_j is front e-linked with some w_i^e.

(4) z_j is back e-linked with some z_v ($z_j < w_i^e \leq z_v$ for some w_i^e).

From lemma 22 it follows that we will find such a stage. In case (1) answer $j \in A_z$ and in (2) negatively. In case (3) say that

$$j \in A_z \iff i \in A_{w^e}.$$

This is true; indeed, since $w_i^e < z_j$ (as we have a front link), we have $j \in A_z \Rightarrow i \in A_{w^e}$. For the other direction we note that this link is not going to be cancelled (by our assumptions about s_0 and lemma 18). So, if i enters A_{w^e} at some later stage, this will be due to a movement of y motivated by some P_t for $t > e$; this means that the link will be visible from P_t and y will follow it, thus pushing j into A_z.

A similar argument (but with z_j in place of w_i^e and vice-versa) shows that in case (4) we also have $j \in A_z \iff i \in A_{w^e}$. □

2.5 Strong reducibilities on $\mathcal{S}_x$.

In this section we demonstrate that strong reducibilities lose some of their generality when restricted to the class of x-sets $\mathcal{S}_x$. In other words the oracle computations are of more special nature, a fact which (as we noted before) allowed us to prove the strong result in section 2.4 (theorem 10) which is not true in the general case (i.e. the whole structure of m-degrees inside a Turing degree). In particular we show that the positive and btt(1) (or $m*$ as we call it, see definition 22 or [23]) coincide with the m-reducibility.

The m-reducibility consists of *one positive* query. We remind the reader of *btt*(1) reducibility, which consists of either one positive or *one negative* query, and which we prefer to call '$m*$ reducibility' (see [23] for more on that). Formally,

Definition 22. *For any two sets A, B we say that $A \leq_{m*} B$ (via f, g) when there are computable functions f, g such that for all n,*

$$n \in A \iff \begin{cases} f(n) \in B & \textit{if } g(n) = 0 \\ f(n) \notin B & \textit{otherwise.} \end{cases}$$

Proposition 5. *If x is non-computable then $\leq_{m*}\restriction \mathcal{S}_x = \leq_m\restriction \mathcal{S}_x$ and in particular, for every two sequences z, w with $\lim z = \lim w = x$ and $A_z \leq_{m*} A_w$ via f, g, $g(n) = 0$ for almost all n. In other words the $m*$-reduction is actually an m-reduction with finitely exceptions (i.e. negative queries).*

Proof. Suppose that for the A_z, A_w above, $A_z \leq_{m*} A_w$ and $g(n) > 0$ for infinitely many n. We will prove that x is computable. Indeed, we can effectively find the zeros of g, so that there is an increasing function h such that for all n, $g(h(n)) > 0$. But then we have

$$h(n) \in A_z \iff f(h(n)) \notin A_w$$

So for every n, $z_{h(n)}$, $w_{h(n)}$ are not on the same side of x. And since h is increasing, these are subsequences of z, w with the property

$$\lim_n |z_{h(n)} - w_{h(n)}| = 0$$

and for all n,

$$x \in (\min\{z_{h(n)}, w_{h(n)}\}, \max\{z_{h(n)}, w_{h(n)}\}).$$

But this means that x is computable, a contradiction. □

We remind that positive reducibility $\leq_p$ is like tt but we don't allow negative queries (formally, the p-formulas are constructed from the atoms via $\{\wedge, \vee\}$ instead of $\{\neg, \wedge, \vee\}$; for more details see [23]).

Proposition 6. *Suppose that $x = \lim z = \lim w$. If $A_z \leq_p A_w$ then $A_z \leq_m A_w$. Moreover, the second reduction is obtained effectively from the first one (given w).*

Proof. Suppose that $\{\sigma_n\}_{n\in\mathbb{N}}$ is an effective enumeration of the positive (p-) conditions (i.e. the propositional formulas built from the atoms $m \in X$ by applying $\vee$, $\wedge$ inductively, using the standard syntactical rules). For the proof of the proposition it is enough to define an algorithm g which takes a number n and (the program for) a computable sequence of rationals w as inputs, and outputs a number $g(n, w)$ such that

$$\sigma_n \vDash A_w \iff g(n, w) \in A_w. \tag{2.15}$$

Indeed, having defined such an effective procedure, suppose that we are given a p-reduction $A_z \leq_p A_w$ via a computable function f, i.e.

$$n \in A_z \iff \sigma_{f(n)} \models A_w.$$

Then for every n we have

$$\sigma_{f(n)} \models A_w \iff g(f(n), w) \in A_w$$

which gives

$$n \in A_z \iff g(f(n), w) \in A_w$$

i.e. an m-reduction (which we got effectively from f and w).

We define the program g by induction on the length[4] of the p-conditions.

For all n and computable sequences of rationals w we define $g(n, w)$ as follows;

1. If $\ell(n) = 0$ then σ_n is an atom, say '$t \in X$' (where t is a number we get effectively from n). For all w define $g(n, w) = t$.

2. Suppose $m > 0$ and $g(t, w) \downarrow$ for all w and t with $\ell(t) < m$. If $\ell(n) = m$ for some formula σ_n, then σ_n is $\sigma_k \vee \sigma_s$ or $\sigma_k \wedge \sigma_s$ for k, s with $\ell(k), \ell(s) < m$. In the first case define

$$g(n, w) = \begin{cases} g(k, w) & \text{if } w_{g(k,w)} \leq w_{g(s,w)} \\ g(s, w) & \text{otherwise} \end{cases}$$

and in the second case define

$$g(n, w) = \begin{cases} g(s, w) & \text{if } w_{g(k,w)} \leq w_{g(s,w)} \\ g(k, w) & \text{otherwise.} \end{cases}$$

To finish the proof, we prove by induction that (2.15) holds for all n, w. If $\ell(n) = 0$ it is obvious. If $\ell(n) > 0$ and for all σ_t with $\ell(t) < \ell(n)$ it holds then two can happen;

1. If $\sigma_n = \sigma_k \vee \sigma_s$ then $\ell(k), \ell(s) < \ell(n)$ and by induction hypothesis

$$\begin{aligned} \sigma_k \models A_w &\iff g(k, w) \in A_w \\ \sigma_s \models A_w &\iff g(s, w) \in A_w \end{aligned}$$

[4] The length ℓ of a p-condition is defined by induction as usual: if σ_n is an atom then $\ell(n) = 0$; and if σ_n is $\sigma_k \vee \sigma_s$ or $\sigma_k \wedge \sigma_s$ then $\ell(n) = \max\{\ell(k), \ell(s)\} + 1$.

for all w. But

$$\sigma_n \models A_w \iff \sigma_k \models A_w \vee \sigma_s \models A_w \iff g(k,w) \in A_w \vee g(s,w) \in A_w$$

for all w. And if for a particular w, $w_{g(k,w)} \leq w_{g(s,w)}$ then

$$g(k,w) \in A_w \vee g(s,w) \in A_w \iff g(k,w) \in A_w$$

(by definition of A_w) which means that (2.15) holds for this w (by definition of g). Also if $w_{g(k,w)} > w_{g(s,w)}$,

$$g(k,w) \in A_w \vee g(s,w) \in A_w \iff g(s,w) \in A_w$$

which means again that (2.15) is correct for this w.

2. If $\sigma_n = \sigma_k \wedge \sigma_s$ then $\ell(k), \ell(s) < \ell(n)$ and by induction hypothesis

$$\begin{aligned} \sigma_k \models A_w &\iff g(k,w) \in A_w \\ \sigma_s \models A_w &\iff g(s,w) \in A_w \end{aligned}$$

for all w. But

$$\sigma_n \models A_w \iff \sigma_k \models A_w \wedge \sigma_s \models A_w \iff g(k,w) \in A_w \wedge g(s,w) \in A_w.$$

And if $w_{g(k,w)} \leq w_{g(s,w)}$ then

$$g(k,w) \in A_w \wedge g(s,w) \in A_w \iff g(s,w) \in A_w$$

(by definition of A_w) which means that (2.15) holds for this w (by definition of g). Also if $w_{g(k,w)} > w_{g(s,w)}$,

$$g(k,w) \in A_w \wedge g(s,w) \in A_w \iff g(k,w) \in A_w$$

which means again that (2.15) is correct.

So in any case (2.15) holds for n and all w, and the induction step is proved.

$\square$

2.6 Immunity properties.

In this section we look at the immunity of the sets A_z given a c.a. real x and sequences z with limit x. As we explained in the introduction, one may think that the more immune the set A_z (or its complement) is, the more complicated the real x is (one can prove that the immunity of such a set does not depend on the choice of z); this is because, in a way, the more immune e.g. the set A_z is, the more difficult is to make correct 'guesses' about terms of z which are on the left of x (so, rationals in the left Dedekind cut of x). However we show that not only the immunity of A_z is independent of z, but it is in a sense independent of x itself! In particular, assuming that x is not computable, A_z cannot be (co-)hh-immune but it is always bi-h-immune or (co-)h-simple (the later when x is semi-computable). So we always have h-immunity and hh-immunity never occurs.

2.6.1 Hyperimmunity.

Suppose that $\lim z = x$ for a computable sequence of rationals $z = \{z_s\}$.

Proposition 7. *If $x = \lim z$ is not c.e. then A_z is h-immune and if it is not co-c.e., then B_z is h-immune. So, if x is not semi-computable, A_z, B_z are bi-immune. Also, if it is c.e. (co-c.e. resp.) non-computable then A_z (B_z resp.) is hypersimple.*

Proof. Suppose that x is not c.e. and A_z was not h-immune. Then there exists a disjoint strong array $D_{g(n)}$ such that for every n,

$$D_{g(n)} \cap A_z \neq \emptyset$$

Now consider the sequence

$$y_s = \min\{z_n \mid n \in D_{g(s)}\}$$

which is a computable sequence of rationals with the property for all s, $y_s < x$. Indeed, if that was not the case for some s, this would mean that $D_{g(s)} \cap A_z = \emptyset$. Moreover, since the array $D_{g(n)}$ is disjoint and $\lim z = x$, it follows that $\lim y = x$. But this is a contradiction since we assumed that x is not c.e. So A_z is in fact h-immune. The case of x co-c.e. can be treated by a dual proof and the rest of the statements in the proposition follow easily. □

After we sorting out h-immunity we would like to look at hh-immunity (and in particular prove that it never occurs). This is more difficult, and we consider separately the cases when x is semi-computable or not.

2.6.2 Non semi-computable reals and hh-immunity.

Theorem 11. *If x is not semi-computable and $z = \{z_s\}$ is a computable sequence of rationals with $\lim z = x$, then A_z, B_z are not hh-immune.*

Proof. Given a set A, define a tree $I(A) : \Sigma^* \to \mathcal{P}(A)$ (where $\mathcal{P}(A)$ is the powerset of A and $\Sigma = \{0, 1\}$). For all $w \in \Sigma^*$ define

$$\begin{aligned} I_\emptyset(A) &= A \\ I_{w0}(A) &= I_w(2A) \\ I_{w1}(A) &= I_w(2A+1) \end{aligned}$$

In this way we split A into the nodes of a tree (which represent subsets of A) such that, if two nodes $I_{w_1}(A)$, $I_{w_2}(A)$ lie in different branches (that is $w_1 \mid w_2$ i.e. they are incomparable w.r.t. the lexicographical ordering of binary strings) then $I_{w_1}(A) \cap I_{w_2}(A) = \emptyset$. We will only need the tree $I(\mathbb{N})$ which we are going to write simply as I in the following. It is easy to see that all nodes of this tree are infinite subsets of $\mathbb{N}$. Now we define a suitable disjoint weak array $W_{g(n)}$ $(n \in \mathbb{N})$ which indicates that A_z is not hh-immune. The computable function g is implicitly defined in the following. The enumeration of $W_{g(n)}$ (for a particular n) is associated with the node I_{1^n0} of the tree I. In particular, it is defined as follows: start enumerating all $s \in I_{1^n0}$ (for successively larger s) and when you come across an s_0 such that z_{s_0} is smaller that all z_s for the s enumerated so far (that is, for all $s \in I_{1^n0}$ with $s < s_0$), enumerate s_0 into $W_{g(n)}$; continue in the same way. Since each node I_{1^n0} is a c.e. set and x is not semi-computable, it is impossible to have $I_{1^n0} \subset A_z$ or $I_{1^n0} \subset B_z$. So, in our enumeration we will find some terms z_s lying on the left of x *and* some lying on the right of x. So, since $\lim z = x$ we have that for all n, $W_{g(n)}$ is finite and the last element s enumerated in it, is in A_s. So

$$W_{g(n)} \cap A_z \neq \emptyset$$

Finally, the array $W_{g(n)}$ is also disjoint since for all n, $W_{g(n)} \subset I_{1^n0}$ and

$$n \neq m \Rightarrow 1^n0 \mid 1^m0 \Rightarrow I_{1^n0} \cap I_{1^m0} = \emptyset$$

Now the array $W_{g(n)}$ with all the above properties witnesses that A_z is not hh-immune. The case for B_z is dual. □

Remark 3. *One other question is for which c.e. reals $x = \lim z$ the set A_z is promptly simple. If the degree of x is not promptly simple, then obviously there is no (computable) sequence z with limit x and A_z promptly simple. Also, it is easy to construct reals $x = \lim z$ such that A_z is promptly simple. The requirements for a basic such construction are:*

$$\begin{aligned} Q_e: &\quad W_e \text{ infinite} \Rightarrow \exists x \exists s \; x \in W_{e,\text{ at } s} \cap A_z[s] \\ P_e: &\quad\quad\quad\quad A_z \neq \varphi_e \end{aligned}$$

and they are satisfied as in a usual finite injury construction (we have restraints for the requirements P_e). Note that instead of requiring $\mathbb{N} - A_z$ to be infinite, we require A_z to be non-computable, which here amounts to the same thing. The requirements Q_e are easily compatible with a large range of other requirements. Also, in a construction where non-computability of A_z is guaranteed by other requirements, P_e may be omitted.

2.6.3 Semi-computable reals and hh-immunity.

According to the above, if x is not semi-computable, both A_z, B_z are h-immune and not hh-immune. On the other hand, if x is c.e. non-computable, A_z is h-simple (if not co-finite); a dual result holds for co-c.e. reals. A natural question is whether A_z or B_z can be hh-immune for semi-computable reals (note that the proof for the case of non semi-computable x cannot be adapted for this case). We prove not only a negative answer to this, but also that A_z or B_z cannot be even finitely strongly h-immune (fsh-immune for short). Before presenting the result, we remind the definition of fsh-immunity.

Definition 23 (Soare[30]). *A set D is finitely strongly h-immune if (it is infinite and) there is no disjoint weak finite array $W_{g(n)}$ (g computable, $W_{g(n)}$ finite for all n, and $n \neq m \Rightarrow W_{g(n)} \cap W_{g(m)} = \emptyset$) all of its members intersecting it and $D \subset \cup_i W_{g(i)}$. In other words, if $W_{g(n)}$ is such an array then $\exists n[W_{g(n)} \cap D = \emptyset]$. D is finitely strongly h-simple (fsh-simple) if it is c.e. and $\mathbb{N} - D$ is fsh-immune.*

Theorem 12. *If x is c.e. then A_z is not fsh-simple for any computable sequence of rationals $z = \{z_s\}$ with $\lim z = x$.*

And according to the previous discussion we have

Corollary 3. *If $x = \lim z$ for $z = \{z_s\}$ computable sequence of rationals, then A_z, B_z are not hh-immune.*

Note that a dual version of theorem 12 holds for co-c.e. reals (by similar proof).

2.6.4 Proof of theorem 12.

The proof is a kind of finite injury construction and is presented in the following sections. Suppose $x = \lim z$ where x is a non-computable c.e. real and A_z is infinite and co-infinite (the other case being trivial). W.l.o.g. we also assume that $z_n \in (0,1)$ for all n; $\{x_s\}$ is an increasing sequence with limit x.

About the construction.

We want to define a weak array $W_{g(t)}$ which shows that B_z is not fsh-immune. The idea is to try to install a sequence of markers y_i and witnesses $w_k = z_{i_k}$ on the right hand side of x so that the following holds:

$$x < \cdots < w_1 < y_1 < w_0 < y_0$$
$$i_k \in W_{g(k)}$$

where g indicates a uniform enumeration of the (indices of the) terms of z into separate 'boxes' $W_{g(k)}$ (and is defined implicitly during the construction). We can have the weak array $W_{g(k)}$ disjoint by ensuring that any element enumerated in $W_{g(k)}$ at a particular stage has not been enumerated in any $W_{g(i)}$ during the earlier stages. Beyond some point, only numbers n with $y_{i+1} < z_n < y_i$ will be enumerated in $W_{g(i)}$ and so we will have $|W_{g(i)}| < \infty$ for all i (since $\lim z = x$); this also helps to succeed $\cup_{t\in\mathbb{N}} W_{g(t)} \supseteq B_z$. Moreover, the witness w_k will ensure that $W_{g(k)} \cap B_z \neq \emptyset$ (since $i_k \in W_{g(k)} \cap B_z$). That $\mathbb{N} - B_z = A_z$ is c.e. it is easy to see.

The difficulty is that since we assume that x is not computable (this case is trivial) it is not easy to find which terms of $\{z_i\}$ lie on the right of x. Also it is not easy to find rationals close to but greater than x. So we have to approximate y_i and w_i by making guesses y_i^s, w_i^s. After finitely many guesses, we will have a suitable y_i^s (and w_i^s) and the construction will not change it later. So we will have $\lim_s y_i^s = y_i, \lim_s w_i^s = w_i$ and the limits are finite (in the sense that after some point the sequence becomes constant). Some of the y_i^s's may lie on the left of x (we say that the *guess is false*) in which case the false guess will be *recovered* (at some later stage s_1) by a fixed (increasing) sequence $\{x_k\}$ which tends to x from the left (i.e. $x_{s_1} > y_i^s$). In this case we say that y_i^s (or y_i) is *injured* (at stage s_1). And according to the construction we leave it undefined; in symbols $y_i^{s+1} \uparrow$. Now we make w_i^s dependent on y_i^s.

Definition 24. *Define*

$$w_i^s = max\{z_t : t \leq s \wedge z_t < y_i^s \wedge t \notin \cup\{W_{g(j),s} : j < s, j \neq i\}\}.$$

If $y_i^s \uparrow$ *then* $w_i^s \uparrow$ *and* $max\{\emptyset\} \uparrow$.

Note that the s in $W_{g(j),s}$ denotes the enumeration into $W_{g(j)}$'s *defined in the construction* and *not* a general enumeration of all c.e. sets. So $t \notin \cup\{W_{g(j),s} : j < s, j \neq i\}$ means that t has not been enumerated in any of $W_{g(j)}$ for $j < s, j \neq i$ by the s-th stage of the construction.

During the construction, if $y_i^s \uparrow$ then this term 'wants to be defined' or, as we say, it requires attention. More generally

Definition 25. *At any stage* $s + 1$ *we say that* y_n^s *requires attention if one of the following holds*

- $y_n^s \uparrow$
- $y_n^s \downarrow$ *and* $x_{s+1} \geq y_n^s$

Note that the second clause means that while y_n^s was defined, at stage $s + 1$ it is injured, i.e. it is found to be a false guess (and thus it must be corrected). We say that y_n^s *receives attention* at stage s if action is being taken at the particular stage for its (re)definition (and this happens according to its priority). Unfortunately, in general we will not be able to (re)define it at once, and it may take several stages. So at the particular stage we *start taking action* for its (re)definition. In order to indicate this (so that later we know that we have started the (re)definition and continue and finish this procedure) we associate with y_i^s a parameter σ_i^s. This is undefined ($\sigma_i^s \uparrow$) when action is not being taken for the satisfaction of y_i^s; and when action is actually being taken, we store in σ_i^s a value relevant to the last stage of its (re)definition which will enable us to continue and eventually finish the procedure. Of course, the (re)definition of y_i^s may be interrupted by an injury of a y_j with higher priority (i.e. $j < i$). In this case we start from zero at a later stage. Finally, when an injury occurs, say y_i is injured, we *initialize* all y_j for $j > i$. This means that we set $y_i^s \uparrow, \sigma_i^s \uparrow$ for all $j > i$ (s is the current stage).

Construction

Stage 0.

Initialize all y_i^s.

Stage $s+1$.

step A Satisfy the following

$$\left.\begin{array}{l} i,j < s+1;\ y_i^s, y_{i+1}^s \downarrow \\ y_{i+1}^s < z_j \le y_i^s \\ j \notin \cup_{t<s+1} W_{g(t),s} \end{array}\right\} \Longrightarrow j \in W_{g(i),s+1} \text{ (enumerate } j \text{ into } W_{g(i)}\text{)}$$

step B Choose the least y_i with $i < s+1$ which requires attention.

$\rightsquigarrow$ *case 1:* $y_i^s \uparrow$ (so $y_j \uparrow$ for $j > i$) and $\sigma_i^s \uparrow$ (i.e. action is not being taken for the redefinition of y_i^s). If $w_{i-1}^{s+1} \uparrow$ do nothing. Otherwise *start taking action* for y_i: Check whether $x_{s+1} + \frac{1}{s+1} < w_{i-1}^{s+1}$. If it is, then define

$$y_i^{s+1} = x_{s+1} + \frac{1}{s+1};\ \sigma_i^{s+1} \uparrow$$

If not, put $y_i^{s+1} \uparrow$ and $\sigma_i^{s+1} = x_{s+1} + \frac{1}{s+1}$.

$\rightsquigarrow$ *case 2:* $y_i^s \downarrow$ and it is *injured* at stage $s+1$, i.e. $x_{s+1} > y_i^s$. First we *initialize* all y_j with $j > i$. If $y_i^s = x_{s_1} + \frac{t}{s_1+k}$, we try whether $x_{s_1} + \frac{t+1}{s_1+k} < w_{i-1}^{s+1}$. If yes, then put

$$y_i^{s+1} = x_{s_1} + \frac{t+1}{s_1+k};\ \sigma_i^{s+1} \uparrow$$

If not, then put $y_i^{s+1} \uparrow$ and $\sigma_i^{s+1} = x_{s_1} + \frac{t+1}{s_1+k}$.

$\rightsquigarrow$ *case 3:* $y_i^s \uparrow$ and $\sigma_i^s \downarrow$ (i.e. y_i^s is undefined *but* action is being taken for its (re)definition). σ_i^s will have the form $x_{s_1} + \frac{t}{k}$. We try whether $x_{s_1} + \frac{t}{k+1} < w_{i-1}^{s+1}$. If yes, then define

$$y_i^{s+1} = x_{s_1} + \frac{t}{k+1}$$

If not, then put $y_i^{s+1} \uparrow$, $\sigma_i^{s+1} = x_{s_1} + \frac{t}{k+1}$.

Verification

The first step is to prove that for all i the following limits exist

$$\lim_s w_i^s = w_i;\ \lim_s y_i^s = y_i;\ \lim_s \sigma_i^s = \uparrow$$

and

$$x < \cdots < y_i < w_{i-1} < y_{i-1} < \cdots < w_0 < y_0 \tag{2.16}$$

We will prove this by induction. Since it is $y_0^s = y_0$ for all s, its enough to prove the induction step described bellow. Our hypothesis is that for all $s > s_0$ and $i < n$ (for some fixed s_0, n) we have

$$w_i^s = w_i;\ y_i^s = y_i;\ \sigma_i^s = \uparrow$$

$$x < w_{n-1} < y_{n-1} < \cdots < w_0 < y_0$$

and we want to find $s_1 > s_0$ such that for all $s > s_1$

$$w_n^s = w_n;\ y_n^s = y_n;\ \sigma_n^s = \uparrow$$

$$x < w_n < y_n < \cdots < w_0 < y_0$$

The proof for y_n.

case 1 $y_n^{s_0} \uparrow;\ \sigma_n^{s_0} \uparrow$.

According to the construction, y_n receives attention at stage $s_0 + 1$ (it has the highest priority). So we check whether $x_{s_0+1} + \frac{1}{s_0+1} < w_{n-1}$.

1A. If true, then we define $y_n^{s_0+1} = x_{s_0+1} + \frac{1}{s_0+1}$.

$1A_1$. If $y_n^{s_0+1} > x$ then for all $s > s_0$, $y_n^{s_0+1} = y_n^s = y_n$.

$1A_2$. If the guess is false and $y_n^{s_0+1} < x$, at some stage $s > s_0 + 1$ the false guess will be recovered, i.e. we will have $x_s > y_n^{s_0+1}$. At that stage (according to the construction) a sequence of corrections will follow of the form

$$x_{s_0+1} + \frac{2}{s_0 + k_2}, x_{s_0+1} + \frac{3}{s_0 + k_3}, \ldots$$

where $k_t = \mu m[\frac{t}{s_0+m} < w_{n-1}]$.

Claim. *This sequence of corrections cannot be infinite and there will be* t, s_1 *such that* $\forall\ s > s_1\ [y_n^{s_1} = x_{s_0+1} + \frac{t}{s_0+k_t} = y_n^s]$.

Proof of claim Suppose not. We know that $x < w_{n-1}$ and $x_{s_0+1} < x < w_{n-1}$, so there are t, t_1 such that $x < x_{s_0+1} + \frac{t}{s_0+t_1} < w_{n-1}$ which implies $x < x_{s_0+1} + \frac{t}{s_0+k_t} < w_{n-1}$. So after trying $y_n^{s_1} = x_{s_0+1} + \frac{t}{s_0+k_t}$ we will have $\lim_s y_n^s = y_n^{s_1}$ (finite limit) which contradicts our hypothesis. $\square$

$1B$. If false, as above, a sequence of corrections will follow of the form $x_{s_0+1} + \frac{t}{s_0+k_t}, t = 1, 2, \ldots$ which must come to an end, so it stops at some $x_{s_0+1} + \frac{t_0}{s_0+k_{t_0}}$. We have $\lim_s y_n^s = y_n = x_{s_0+1} + \frac{t_0}{s_0+k_{t_0}}$.

case 2 $y_n^s \uparrow$; $\sigma_n^{s_0} \downarrow$.
The case is similar to $1B$ above and $\lim_s y_n^s$ exists and is less than w_{i-1}.

case 3 $y_n^{s_0} \downarrow$ (so $\sigma_n^{s_0} \uparrow$) and it is injured at stage s_0 (or at a later stage). Then, in the same way as in $1A_2$ above, $\exists \lim_s y_n^s < w_{i-1}$. Of course if it is not injured later, the same result is obvious.

The proof for w_n**.** The crucial point is to prove $\exists \lim_s w_n^s = w_n$. Suppose that $\forall\ s > s_1\ y_n^{s_1} = y_n$ (s_1 is the least such). Then according to the construction $y_{n+1}^{s_1+1} \uparrow$ and it remains so all the time (i.e. stages $s > s_1$) that $w_n^s \uparrow$ (and of course $y_i^s \uparrow$ for $i > n, s > s_1$ such that $w_n^s \uparrow$).

First we notice that w_n^s will be defined at some stage $s > s_1$ because infinitely many terms z_j will appear at subsequent stages in a close area of x, which have not been enumerated in any $W_{g(i)}$.

if $w_n^s > x$ then $w_n^s < y_n$ and $\exists \lim_t w_n^t$ since there are finitely many z_i's between w_n^s and x (and w_n^s is non-decreasing on s).

if $w_n^s < x$ then we will show that at some point w_n^s will be redefined to $w_n^t > x$. Indeed, if not, at the stages $s > s_1$ such that $y_{n+1}^s \downarrow$ we have $y_{n+1}^s < w_n^s < x < y_n^s$ and so no z_i ($i > s_1$) with $x < z_i < y_n^s$ appears in such intervals (i.e. if $s_2 < i < s_3$ and $\forall s\ [s_2 < s < s_3 \Rightarrow y_{n+1}^s \downarrow]$ then $z_s \notin (x, y_n^s)$ for any $s \in (s_2, s_3)$). And at intervals (s_4, s_5) such that $\forall s \in (s_4, s_5)\ y_{n+1}^s \uparrow$ we have $z_s \notin (x, y_n^s)$ because otherwise w_n^s would be redefined to an element $> x$. So by induction it follows that $\forall s > s_1\ z_s < x$ which contradicts our hypothesis that $\exists^\infty s\ z_s > x$. So eventually for some $s > s_1$ we have $y_n^s > w_n^s > x$.

And from that point w_n^s will be non-decreasing, and so the sequence $\{w_n^s\}_s$ will become eventually constant, reaching a final $w_n < y_n$ (since $\lim z = x$).

The rest of the verification. Finally from the construction it is easy to see that $i \neq j \Rightarrow W_{g(i)} \cap W_{g(j)} = \emptyset$ and $\forall i\ |W_{g(i)}| < \infty$ (the last because after y_i, y_{i+1} are fixed, only finitely many terms of $\{z_j\}$ can appear in (y_{i+1}, y_i)). It is also clear that $w_j \in W_{g(j)}$ and $\exists \lim_s \sigma_i^s =\uparrow$. Finally $\cup_{i\in\mathbb{N}} W_{g(i)} \supseteq B_z$ since $\lim_s y_s = x$ (due to the convergence of $\{z_s\}$ and (2.16)) and for any i, after y_i, y_{i+1} are fixed, all terms of $\{z_s\}$ that will appear in (y_{i+1}, y_i) will have their indices in $W_{g(i)}$.

Chapter 3

Approximation Representations for Δ_2 Reals

in

3.1 Introduction

There are many ways to study real numbers from an effectiveness point of view. Most of the work has been done in classical computability theory (see Odifreddi [23, 24]) and in particular the study of degree structures and hierarchies of reals (i.e. sets). Other work is on randomness, see e.g. [1]. Another approach is concerned principally with what ways (in some sense related to effectiveness) a real number can be approximated by a sequence of rationals. This approach is more in the framework of *computable analysis*, see e.g. Calude, Coles, Hertling, Khoussainov[9], Calude, Hertling[10] and Rettinger, Zheng et al. [27], Zheng[36] for hierarchies of reals. In chapter 2 we initiated the study of Δ_2 reals x by means of the structure of the sets

$$A_z = \{i \mid z_i < x\}$$

where z is a computable sequence of rationals (z_i) with limit x, under strong reducibilities. It is well known that the limits of a computable sequences of rationals are exactly the Δ_2 reals, and so our approach is restricted to this important class, the reals T-reducible to $\mathbf{0}'$. In fact we are only interested in sets A_z that are bi-infinite (something we assume from now on). In this case we call z a *symmetric approximation* to x.

Definition 26. *If* $\lim z = x$ *is a symmetric approximation to* x*, the set* A_z *is called an approximation representation of* x*.*

We often say just *representation* for short. It is clear that such a set *represents* a particular computable approximation of a real. We have a correspondence of a Δ_2 real with all its representations and conversely a representation may be a representation of many different reals; see figure 3.1.

Figure 3.1: Reals and representations

A basic fact is

Proposition 8. *(see chapter 2) Every representation of a real is Turing equivalent with it.*

So all representations of a real lie in the same T-degree and that is why we are interested in strong reducibilities $\leq_{\mathrm{r}}$. Under $\leq_{\mathrm{r}}$ the r-degrees of representations of a real x form a substructure $\mathcal{D}_x^{\mathrm{r}}$ of the r-degrees within the Turing degree of x. We call this the *approximation structure* of x.

Moreover, a representation of x is c.e. iff x is c.e. (i.e. the limit of a computable increasing sequence of rationals). The main results in chapter 2 were about c.e. reals, and so all the representations we considered were c.e. We constructed a c.e. x such that an infinite antichain is embeddable in $\mathcal{D}_x^{\mathrm{wtt}}$. The same method can be used to embed an infinite computably independent set of sets $\{A_i\}$ (i.e. with $A_n \not\leq_{\mathrm{wtt}} \oplus_{i\neq n} A_i$) and so construct an x such that every countable partial ordering is embeddable in $\mathcal{D}_x^{\mathrm{wtt}}$. In contrast we constructed a non-computable c.e. x such that all of its representations are m-equivalent (i.e. $\mathcal{D}_x^{\mathrm{m}}$ consists of a single element).

By exploring the variety of representations that a c.e. (and in general Δ_2) real can have, from a computational point of view (e.g. strong reducibilities), we aim at a classification of these reals according to their approximation properties. This approach is natural since approximation is a characterising feature of the Δ_2 class. Also it is a different way of looking at those reals, and we very much like to establish connections with existing classifications.

In this chapter we continue in the line of chapter 2 but looking at more advanced questions. In the next section we give a characterisation of representations in terms of cuts or linear orderings. We show that representations are exactly the cuts of computable orderings of $\mathbb{N}$, of order type $\omega + \omega^*$. So a Δ_2 real naturally defines a class of such cuts (i.e. its representations) and most of the results below can be stated in terms of cuts. In the same section we also mention that no representation lies on a proper class of the difference hierarchy and that there are reals that have different wtt-degree than any of their representations.

In section 3.3 we look at the question of how the representations of two T-equivalent reals are related. We construct T-complete x_1, x_2 and a representation A of x_1 such that every representation of x_2 is wtt-incomparable to A. So the two structures are not necessarily related computationally (appart from the fact that they lie in the same T-degree). The proof uses an infinite injury argument, and it is the first one we use for the construction of representations.

In section 3.4 we look at density: given $A_1 <_{\text{wtt}} A_2$ representations of a c.e. x is there a representation of it A with $A_1 <_{\text{wtt}} A <_{\text{wtt}} A_2$? Although the wtt-degrees of c.e. sets are dense, it turns out that a negative answer is true. We use an infinite injury tree argument to construct suitable x, A_1, A_2 and support this claim.

We assume that the reader is familiar with computability theory and in particular with priority arguments on a tree. We follow the standard notation in computability theory. For background in computable analysis the references given above are useful. The results in this chapter are published in [5].

3.2 Some facts about representations

Let (z_n) be a computable (say injective) sequence of rationals. This sequence defines a computable linear ordering $\prec_z$ on $\mathbb{N}$: $n \prec_z m \iff z_n < z_m$. If (z_n) converges symmetrically to some x, A_z is a bi-infinite cut[1] of that computable ordering. It is known (see Odifreddi[23]) that the cuts of computable linear orderings of $\mathbb{N}$ are exactly the semi-recursive sets. We recall the following

Definition 27 (Jockusch). *A set A is semirecursive if there is a computable f such that*

- $f(x, y) \in \{x, y\}$

[1] *cut* of a linear ordering $<$ of $\mathbb{N}$ is a downwards or upwards $<$-closed subset of $\mathbb{N}$. We often identify a cut with its complement.

- $x \in A \;\vee\; y \in A \Rightarrow f(x,y) \in A$.

So approximation representations are semi-recursive sets, but the converse doesn't hold, as the following proposition shows. Let ω^* be the inverse of the usual ordering ω of the naturals. It is not difficult to show that *any linear ordering of* $\mathbb{N}$ *in which every element has either finitely many predecessors or finitely many successors, is isomorphic to* $\omega+\omega^*$. Also, *any linear ordering of* $\mathbb{N}$ *which has a unique bi-infinite cut is isomorphic to* $\omega+\omega^*$. We also have

Proposition 9. *A set of naturals is an approximation representation of some* Δ_2 *real iff it is the bi-infinite cut of a computable linear ordering of* $\mathbb{N}$*, of order type* $\omega+\omega^*$.

Proof. Suppose we are given such an ordering $\prec$ and C its unique bi-infinite left cut; we will define a (symmetrically) convergent computable sequence z such that $A_z = C$. We define z_0 in the middle of $(0,1)$ and suppose that for all $i < s$, $z_i \downarrow$. Let a_s be the largest z-term with index $< s$ and $\prec s$; and $a_s = 0$ if such doesn't exist. Also let b_s be the smallest z-term with index $< s$ and $\succ s$; and $b_s = 1$ if such doesn't exist. Then define z_s in the middle of (a_s, b_s).

Since $\prec$ is computable, the definition of z is effective and z is computable. By induction $s \prec n \iff z_s < z_n$. We prove that z converges. For every $s \in C$ there are only finitely many $n \prec s$. So there must be an $s_1 \in C$ with $z_s < z_{s_1}$. So there is an increasing sequence (s_i) of elements in C such that (z_{s_i}) is increasing and so converging, say to a. Dual observations hold for $\overline{C}$ and so we get a decreasing and converging (say to b) (z_{n_i}) with $n_i \in \overline{C}$. It is enough to show that $a = b$. Indeed, otherwise the interval (a,b) would be proper and no term of the two sequences would appear in it. But according to the way we define z any large enough z_s will appear in (a,b), a contradiction.

So z converges and since A_z is a bi-infinite cut, it is identical to C. Finally it is obvious that an approximation representation A_z is a cut of a computable ordering of type $\omega+\omega^*$, which concludes the proof. □

Another interesting question is how a representation of a real x relates to x w.r.t. strong reducibilities. We observe

Proposition 10. *No wtt-complete c.e. real* x *has a representation* $\equiv_{wtt} x$.

Proof. In chapter 2 we showed that any representation A_z of x is a hypersimple set. And it is known that no such sets are wtt-complete. □

Finally we note

Proposition 11. *Representations are either c.e. or co-c.e. or they don't belong to any finite level of difference hierarchy.*

To see this, first note that if A_z is not c.e. or co-c.e. then it is bi-immune (see chapter 2). Then the proposition follows from

Lemma 24. *No set in a finite level of the difference hierarchy is bi-immune.*

Proof. Suppose that A is properly n-c.e. and n is even. We show that $\overline{A}$ is not immune. Let $\lim_s \phi(m,s) = A(m)$, $\phi(m,0) = 0$ and

$$|s \ : \ \phi(m,s) \neq \phi(m,s+1)| \leq n \tag{3.1}$$

for every m. Note that since A is properly n-c.e. (so not $(n-1)$-c.e.) (3.1) holds with equality for infinitely many m. To effectively generate an infinite subset of $\overline{A}$ we start looking for k on which ϕ changes exactly n times. We will find infinitely many such k and since n is even they must have $\lim_s \phi(k,s) = 0$ and so belong to $\overline{A}$. The case 'n-odd' is dual (showing that A is not immune). □

3.3 Two approximation structures in $\mathbf{0}'$

It is natural to ask what is the relation of the information content of a real and the variety of its representations. The following theorem shows that reals with the same information content may have quite unrelated approximation structures. This means that a classification of the Δ_2 reals based on their approximation structures is qualitatively quite different from classifications based on information content.

Theorem 13. *There exist Turing complete c.e. reals x_1, x_2 and a representation A_{z^1} of x_1 such that every representation of x_2 is wtt-incomparable with A_{z^1}.*

We will build x_1, x_2 by an approximation procedure, in the framework developed in chapter 2; we review it briefly. For the construction of a c.e. real x with requirements on its representations, we start defining the terms of a sequence z in decreasing order. On the other hand we have a non-decreasing sequence y which controls the enumeration in A_z, i.e. whenever we wish to enumerate $n \searrow A_z$ (say at s) we define $y_s = z_n$. All this action takes place within $(0,1)$ and we picture $(0, y_s)$ as the *black area* (see figure 3.2) which expands, but also tends to a limit (since y is bounded). Also, we always define the z-terms outside the black area (though they may enter it later on).

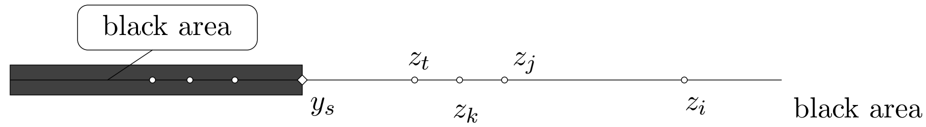

Figure 3.2: Construction

The convergence of z is guaranteed once we make sure that it steadily approaches $\lim y$ (at every stage s an interval (y_s, t) is suggested as appropriate for the definition of z_s; we always define it in the middle of the suggested interval). Eventually we will have $\lim y = \lim z$ and this c.e. real will satisfy the requirements.

The construction of z is most importantly a construction of a computable ordering of $\mathbb{N}$ which admits a unique bi-infinite cut. The properties of this ordering guarantee the satisfaction of the requirements.

The reals x_1, x_2 of theorem 13 will be constructed in a variation of this general framework. We lay out the requirements.

$$\begin{aligned}
\mathcal{R}: &\quad \lim y^1 = \lim z^1 := x_1 \ \&\ \lim y^2 = \lim z^2 := x_2 \\
\mathcal{P}: &\quad K \leq_T x_1 \ \&\ K \leq_T x_2 \\
\mathcal{Q}_e: &\quad \lim w^e = x_2 \Rightarrow \neg[A_{w^e} = \Phi_e^{A_{z^1}}; \phi_e] \ \vee\ A_{w^e} \text{ co-finite} \\
\mathcal{N}_e: &\quad \lim w^e = x_2 \Rightarrow \neg[A_{z^1} = \Phi_e^{A_{w^e}}; \phi_e]
\end{aligned}$$

where Φ_e, ϕ_e are effective enumerations of partial computable functionals and functions respectively and the expression $A = \Phi_B; \phi$ means that the use in these computations is bounded by ϕ (witnessing a wtt-reduction). $\mathcal{R}$ will be sorted out by the framework of the construction, as described above.

3.3.1 $\mathcal{P}$-requirements

To satisfy $\mathcal{P}$ we have to code K into x_1, x_2 *or* into sets equivalent with them. The easiest choice is to code it into A_{z^1}, A_{z^2} since these sets are directly involved in the construction (remember that any representation of a real *is* T-equivalent with it). Notice that the construction z^2 only makes the coding in $\mathcal{P}$ easier; z^2 is not involved in other requirements.

One can see that, if we are to satisfy all requirements, the coding in $\mathcal{P}$ will yield no stronger than T-reduction (i.e. *wtt*-reduction is not possible). Thus we enumerate Turing functionals Γ_1, Γ_2 such that

$$\Gamma_1^{A_{z^1}} = K \ \& \ \Gamma_2^{A_{z^2}} = K.$$

The uses γ_i will be increasing. They will always be defined on elements currently outside A_{z^i} and eventually rest on such an element. Also, at any stage

$$t < k \iff z^i_{\gamma_i(k)} < z^i_{\gamma_i(t)}.$$

3.3.2 $\mathcal{Q}$-requirements

The requirements most difficult to satisfy are the $\mathcal{Q}$ ones; these will bring an infinite injury character to the construction. The difficulty is that we don't have any control on the witnesses w_n, which can be enumerated without our will. The effective list of computable sequences w^e contains many inappropriate ones that we should reject in the first place, were we able to distinguish them in a computable way. Such are e.g. w's with A_w co-finite. For these w's our module will run forever, and we have to ensure that this feature does not harm other requirements (especially $\mathcal{P}$). Here is a strategy for $\mathcal{Q}$.

1. Pick the least unused witness $n \notin A_w$ such that $w_n \downarrow < z^2_{\gamma_2(e+1)}$. *If $\gamma_2(e+1)$ changes during this cycle, $\mathcal{Q}$ is initialized and we start from (1).*

2. Wait until $\Phi^{A_{z^1}}(n) \downarrow = 0; \phi(n) \downarrow$. *If in the meantime $n \searrow A_w$, go to (1).*

3. Let k be the maximum such that $w_n < z^2_{\gamma_2(k)}$. If $z^1_t < z^1_{\gamma_1(k+1)}$ for some $t < \phi(n)$, $t \notin A_{z^1}$, define $y^1_s := z^1_{\gamma_1(k+1)}$.

   We ensure that the Γ_1-markers that sit (on z^1-terms) on the left of w are as many as the Γ_1-markers that sit on the left of z^2 terms involved in the use $\phi(n)$ and yet in the black area.

4. Wait until $\Phi^{A_{z^1}}(n) \downarrow = 0; \phi(n)$ is restored. *(If in the meantime $n \searrow A_w$ go to (1).)*

   Then put $n \searrow A_w$ by defining $y^2_s := w_n$.

5. If $\Phi^{A_{z^1}}(n) \downarrow$ is spoilt, go to (1).

As usual, s denotes the current stage of the construction in which this module works.

Analysis of outcomes The *finite outcomes* are

- Stuck in (1)
- Stuck in (2) or infinitely many visits to (1), (2) but finitely many on the other steps *(count this as finite since no action is taken in the first two steps.)*
- Stuck in (4)
- Stuck in (5)

Note that each of these outcomes is not only successful for $\mathcal{Q}$ but also mean that $\mathcal{Q}$'s module stops interfering with the rest of the construction, from some point on. In particular it allows $\mathcal{P}$ to succeed (since it agitates each Φ-marker for only a finite time). Also any number of $\mathcal{Q}$'s can work together since $\mathcal{Q}_e$ can only agitate $\gamma_i(n)$ for $n > e + 1$.

The *infinite outcomes* are

(a) We pass infinitely often from (3), (4) but only finitely often from (5). (*that is when almost every time we visit (4), the unwanted enumeration happens while waiting for $\Phi^{A_{z^1}}(n)\downarrow$.)*

(b) We reach and leave (5) infinitely often (because of a $A_{z^1} \upharpoonright \phi(n)$ enumeration.

The action involved in the infinite outcomes is *expansion of the z^i-black area.* In case (a) we only have expansion of the z^1-area while in case (b) expansion of both ones. This could interfere with $\mathcal{P}$ or even with other requirements. The idea for showing that it doesn't is to show that these actions, although apparently forced by steps (3),(4), they would anyway occur (sooner or later) by a $\mathcal{P}$-related action. Indeed, if for example we reach (5) and the computation is spoilt, this would be due to a $K \upharpoonright (k+1)$ enumeration. So even if we hadn't act under (3) or (4), this expansion of the black area would happen at the time of the $K \upharpoonright (k+1)$ enumeration; our actions are in acordance with $\mathcal{P}$. This way the impact $\mathcal{Q}$ has in the construction under an infinite outcome (given $\mathcal{P}$) is very little (namely it only affects the timing of the actions and not the actions themselves).

To illustrate this we prove the satisfaction of a single $\mathcal{Q}_e$ and $\mathcal{P}$ in a construction motivated only by these two requirements, and $\mathcal{Q}$ has an infinite outcome.

First we show that all Γ_i-markers eventually rest on $\overline{A_{z^i}}$ (i.e. outside the black area). Note that $\gamma_i(n)$ for $n \leq e+1$ won't be agitated by $\mathcal{Q}_e$. Now by induction: assume that for all $n < n_0$, $\gamma_i(n)$ eventually rest (say after stage s_0). From s_0 all of our w-witnesses will sit on the left of $z^2_{\gamma_2(n_0)}$; indeed, otherwise the module would terminate since the markers on the left of w are stable. So $\gamma_2(n_0)$ eventually rests on $\overline{A_{z^2}}$. According to step (3), $z^1_{\gamma_2(n_0)}$ will not be agitated again (so $\gamma_1(n_0)$ eventually rests).

Now the satisfaction of $\mathcal{Q}$ is evident, once we realize that supposing $\lim w = \lim y$ we get that either the module terminates or A_w co-finite. Indeed, if this didn't hold we would have infinitely many terms on the right of $z^2_{\gamma_2(e+1)}$; but since $z^2_{\gamma_2(e+2)} < z^2_{\gamma_2(e+1)}$ and sit outside the black area, this would contradict $\lim w = \lim y$.

We note that any number of $\mathcal{Q}$-requirements with any outcomes work well along with $\mathcal{P}$ and their satisfaction can be proved inductively as above. In particular no nesting of strategies is needed.

3.3.3 $\mathcal{N}$-requirements

$\mathcal{N}_e$ is easier to satisfy. After finitely many attempts we can ensure that our witnesses stay out of the black area as long as we want. This involves placing any witness z^1_t in a safe position, namely between $z^1_{\gamma_1(e)}$ and $z^1_{\gamma_1(e+1)}$. This will not cause any problems in the construction since we only work on $\mathcal{N}_e$ finitely often.

1. Pick t big (so z^1_t currently undefined) and declare z^1_t witness (so give instruction for z^1_t's definition that $z^1_{\gamma_1(e+1)} < z^1_t < z^1_{\gamma_1(e)}$).

   Wait until $\Phi^{A_w}(t)\downarrow= 0; \phi(t)\downarrow$. If in the meantime $t \searrow A_{z^1}$ or $\gamma_1(e+1)$ changes, start anew.

2. If there are w_i, $i < \phi(t)$ outside the black area with $w_i < z^2_{\gamma_2(e)}$, put all these $i \searrow A_w$ (by defining $y^2_s := w_i$ where w_i is the maximum such w-term).

   Wait until $\Phi^{A_w}(t)\downarrow= 0; \phi(t)$ is restored. If in the meantime $\gamma_1(e+1)$ changes (and so $t \searrow A_{z^1}$), go to (1).

3. Put $t \searrow A_{z^1}$ (by defining $y^1_s := z^1_t$).

4. If $\Phi^{A_w}(t)\downarrow= 0$ is spoilt, go to (1).

Note that the finiteness and success of this module depends solely on the success of $\mathcal{P}$ (that the Γ_1-markers eventually rest). As $\mathcal{Q}$-requirements respect $\mathcal{P}$ and $\mathcal{N}$ do as well (because $\mathcal{N}_e$ doesn't agitate $\gamma_i(n)$ for $n \leq e$) all strategies are compatible.

3.3.4 Construction and Verification

Let us divide $\mathcal{P}$ into $\mathcal{P}_1, \mathcal{P}_2, \dots$ where $\mathcal{P}_n$ denotes the requirement that $\gamma_1(n)$, $\gamma_2(n)$ both eventually rest (and of course Γ_1, Γ_2 hold correct computations). We agree on the following priority list of requirements:

$$\mathcal{P}_0 > \mathcal{N}_0 > \mathcal{Q}_0 > \mathcal{P}_1 > \mathcal{N}_1 > \dots$$

We also assume a uniform numbering of the requirements in this list, so that we can talk about the *i-th requirement* regardless its nature.

At each stage we enumerate one axiom for each Γ_i: find the least t such that $\Gamma_i^{A_{z^i}}(t) \uparrow$ and enumerate the axiom $\Gamma_i^{A_{z^i}}(t) = K(t)$ with big use $\gamma_i(t)$.

At stage $s+1$ we define z_s^i between y_s and the largest z_t^i, $t \leq s$ which lies outside the black area *unless* z_t is subject to a condition set by an $\mathcal{N}$-requirement. In the latter case we define it according to the condition. Note that we only specify where a term should be placed in relation with other defined terms. To make the construction definite, let the definitions be on the middle of the suggested interval.

At $s+1$ we also define y_{s+1}^i after a series of substages. At substage n we run the n-th strategy and get a temporary definition $y_{s+1}^i[n]$ of y_{s+1}^i. We do this for all $n \leq s$ and eventually define $y_{s+1}^i := y_{s+1}^i[s]$.

This concludes the construction but few explanatory words are appropriate. Every time we visit a strategy, we start from where we last stopped. Also the parameters we use have current value, as this was left by the last substage of the current stage (this also applies to the black area). Of course, in order to run a strategy, all parameters mentioned must be defined (otherwise we don't do anything more than deliver the parameters as we got them from the previous strategy, to the next one). Finally if we set a condition z_t^1 according to $\mathcal{N}_e$ and $\gamma_1(e+1)$ changes before $z_t^1 \downarrow$, we remove the condition since it was based on a value that changed.

Verification We proceed inductively, supposing that for all $j < n$, $\mathcal{P}_j$, $\mathcal{N}_j$, $\mathcal{Q}_j$ are satisfied and the ones with finite outcome (including $\mathcal{P}_j$) have stopped acting after s_0. The construction carries on defining $\gamma_i(n)$ and $z^i_{\gamma_i(n)}$ outside the black area. And since $\mathcal{P}_j$, $\mathcal{N}_j$, $\mathcal{Q}_j$ for $j \geq n$ never force $\gamma_i(n) \searrow A_{z^i}$, $\gamma_i(n) \uparrow$ can only happen due to $\mathcal{N}_j$, $\mathcal{Q}_j$ for $j < n$ (given that $\gamma_i(j)$, $j < n$ have stabilised). Since $\mathcal{N}_j$, $j < n$ have ceased to act, they can't be responsible for $\gamma_i(n) \uparrow$ and the same holds for the $\mathcal{Q}_j$, $j < n$ with finite outcome.

Now we can prove that once $z^2_{\gamma_2(n)}$ is defined after s_0, it will stay outside the black area forever. Indeed, otherwise a $\mathcal{Q}_j$, $j < n$ with infinite outcome would come to a witness $w^j_t > z^2_{\gamma_2(n)}$, enumerate $t \searrow A_{w^i}$ under step (4) and hold $\Phi^{A_{z^1}}(t) \neq A_{w^i}(t)$ with use $A_{z^1} \upharpoonright \phi(t)$ that can change only if one of $\gamma_1(k), k < n$ moves (due to the preliminary action of step (3)). By inductive hypothesis the disagreement would be preserved and $\mathcal{Q}_j$ would have finite action, contradiction.

So $z^2_{\gamma_2(n)}$ will eventually rest outside the black area and, according to the above,, no infinitary $\mathcal{Q}$ will pick a w-witness greater than $z^2_{\gamma_2(n)}$. Hence, according to step (3), no such requirement will move $\gamma_1(n)$. And due to the choice of s_0, no other requirement will agitate $\gamma_1(n)$, which will eventually stabilise, giving the success of $\mathcal{P}_n$.

Turning into $\mathcal{N}_n$, let $s_1 > s_0$ be large enough so that $\gamma_i(n)$ have stabilised. No lower priority requirement than $\mathcal{N}_n$ can enumerate $z^1_{\gamma_1(n+1)}$, and so an $\mathcal{N}_n$-witness z^1_t. Thus, only an infinitary higher $\mathcal{Q}_j$ could do that, under step (3) of its module. But again, if this happened we could show that $\mathcal{Q}_j$ has finite outcome: the witness it would hold when performing this enumeration would be greater than $z^2_{\gamma_2(n+1)}$ (otherwise it wouldn't enumerate $z^1_{\gamma_1(n+1)}$). So when it reached (5) (and it will reach it since s_1 is big enough), the computation would be preserved due to the choice of s_1, and the module would terminate; contradiction. Now if $\mathcal{N}_n$ doesn't reach (3), we're done. Otherwise the disagreement will be preserved due to the action in (2) and the choice of s_1.

As far as $\mathcal{Q}_n$ is concerned, if it gets stuck on a step of its module, it is obviously satisfied (as explained when we analysed its outcomes). Otherwise we will have

$$t \in \overline{A_{w^n}} \Rightarrow w^n_t > z^2_{\gamma_2(n+1)} \tag{3.2}$$

for all t. This concludes the induction step in an argument that shows the satisfaction of all $\mathcal{P}, \mathcal{N}$, and the $\mathcal{Q}$ with finite outcome. For the $\mathcal{Q}$ with infinite outcome it shows that (3.2) holds. Now we can see that these are also satisfied; indeed, supposing $\lim y^2 = \lim w^n$ we can see that there are only finitely many terms $w^n_t > z^2_{\gamma_2(n+1)}$ which also means that $\mathcal{Q}_n$ is satisfied. That is because the interval $(z^2_{\gamma_2(n+2)}, z^2_{\gamma_2(n+1)})$ is non-empty and lies outside the black area. So in this case $\overline{A_{w^n}}$ is co-finite, which is what we wanted.

The only thing left to complete the verification is to show that $\mathcal{R}$ is satisfied. Fix $i \in \{1, 2\}$. According to the way that the terms of z^i are defined by the construction, it is enough to prove that

$$\lim y^i = \lim_s z^i_{\gamma_i(s)}. \tag{3.3}$$

Indeed, if we fix an s, almost all z^i-terms will be defined on the left of $z^i_{\gamma_i(s)}$ (because only finitely many terms which carry $\mathcal{N}$-conditions can be defined on the right of it). Now we will use the fact that we define z^i-terms in the middle of the suggested interval. The sequence $(z^i_{\gamma_i(s)})$ is decreasing and bounded; so $\lim_s z^i_{\gamma_i(s)}$ exists and is $\leq \lim y$ (as all of its terms are). Let $\lim y = x$ and consider the sequence recursively defined as

$$\begin{aligned} a_1 &= z^i_{\gamma_i(1)} \\ a_{s+1} &= x + \frac{a_s - x}{2} \end{aligned}$$

(intuitively, we start from $z^i_{\gamma_i(1)}$ and define the next term in the middle of the interval between x and the last term). It is straightforward that $\lim_s a_s = x$. If we prove that

$$a_s \geq z^i_{\gamma_i(s)}$$

for all s, using the fact that $z^i_{\gamma_i(s)} \geq x$ for all s, we get (3.3), i.e. what we need to finish. We prove it inductively: for $s = 1$ it is evident. Suppose that it holds for s. Note that when $z^i_{\gamma_i(s+1)}$ is defined, $z^i_{\gamma_i(s)}$ is already defined and so

$$z^i_{\gamma_i(s+1)} = y_t + \frac{y_t - z_k}{2}$$

for some t, k, with $z_k \leq z^i_{\gamma_i(s)}$ and $y_t \leq x$. By the induction hypothesis, we also have $z_k \leq a_s$, and so

$$z^i_{\gamma_i(s+1)} \leq x + \frac{x - a_s}{2} = a_{s+1}$$

and we are done.

3.4 Non-density of representations

It is natural to ask whether the wtt-degrees of representations of a fixed c.e. real are dense. The following theorem says that this is not always the case, and it is not obvious if we consider that the structure of wtt-degrees of c.e. sets in general is dense.

Theorem 14. *There are c.e. reals y such that the wtt-degrees of the representations of y are not dense.*

We wish to construct two sequences z, x with the same limit and such that $A_z <_{\mathrm{wtt}} A_x$ and for every sequence w with the same limit and $A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x$, either $A_w \equiv_{\mathrm{wtt}} A_z$ or $A_w \equiv_{\mathrm{wtt}} A_x$.

An easy way to code A_z into A_x is to define each z-term on some x-term. This is what we'll do, and note that it implies $A_z \leq_{\mathrm{m}} A_x$.

The density requirement is the hardest, and we will split it into three. Given w, our first attempt will be to try to prevent $A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x$. For the first inequality we have $\mathcal{N}$, and $\mathcal{M}$ will work on preventing the second. If one of them fails to block the inequality, it will produce a certain infinitary outcome about w in relation with x or z. If they *both fail*, the information they give about w in relation with z *and* x, along with the work of a third requirement $\mathcal{Q}$ will deliver $A_w \equiv_{\mathrm{wtt}} A_z$ or $A_w \equiv_{\mathrm{wtt}} A_x$.

Along these lines we now formulate $\mathcal{N}$, $\mathcal{M}$. The usual way to block a wtt-inequality between representations (say $A_z \leq_{\mathrm{wtt}} A_w$) is to pick a witness z_i and wait until $\Phi^{A_w}(i) \downarrow ; \phi$ (where Φ is a possible reduction). Then expand the black area up to the largest w-term less than z_i and wait until the computation is restored. If this happens, the use will be the same, and so no w-term in the use will be outside the black area and less than z_i. this means that now we can expand the black area up to z_i (thus diagonalising) and the computation will be preserved *unless a w-term below the use sits on z_i*.

So $\mathcal{N}$ will block $\leq_{\mathrm{wtt}}$ unless all of its z-witnesses sit on w-terms. And if we try as witnesses a cofinite subset of $\mathbb{N} - A_x$ (we have to employ witnesses that sit outside the black area), failing to block $\leq_{\mathrm{wtt}}$ will produce the outcome that almost all z-terms (outside the black area) sit on w-ones. Similarly, if $\mathcal{M}$ fails to block $A_w \leq_{\mathrm{wtt}} A_x$, this will be because almost every w-term (outside the black area) sits on an x-one.

3.4.1 Requirements

To formalise these ideas, let Z be the set of the indices of the x-terms that happen to sit on z-terms (we know that every z-term is made to sit on an x-term). Similarly, with respect to the given w, let W be the set of indices of the x-terms that happen to sit on w-terms. Then we have

$$\mathcal{N}_w : A_z \leq_{\mathrm{wtt}} A_w \Rightarrow Z \cap \overline{A_x} \subseteq_* W$$

where $\subseteq_*$ means subset modulo finite sets. Now let X_w be the set of indices of w-terms that sit on x-ones. Note that this is a c.e. set, as well as Z, W that we considered above. Then we require

$$\mathcal{M}_w : A_w \leq_{\mathrm{wtt}} A_x \Rightarrow \overline{A_w} \subseteq_* X_w.$$

From the above it is clear that we are working modulo the black area. This means that we are only interested in elements sitting outside of it. This will continue to hold throughout the proof, since for the elements in the black area we can decide their luck by waiting long enough to appear there.

$\mathcal{Q}$-requirements

If both $\mathcal{N}, \mathcal{M}$ are satisfied by their second clause, we know that modulo (i.e. ignoring) the black area, almost every z-term sits on a w-term and almost every w-term sits on an x-term. The job of $\mathcal{Q}$ is to give Z a certain maximality property, but only modulo the black area. Indeed, it is not difficult to show that if Z were maximal then $A_x \leq_{\mathrm{m}} A_z$ [2] and so there is no hope for the requirement $A_z <_{\mathrm{wtt}} A_x$ to be satisfied. Given a sequence w as before, we want

$$\overline{A_x} \cap Z \subseteq_* \overline{A_x} \cap W \Rightarrow \overline{A_x} \cap Z =_* \overline{A_x} \cap W \ \vee \ \overline{A_x} \cap W =_* \overline{A_x}. \tag{3.4}$$

where W comes from w as before and $=_*$ is equality modulo finite sets. Note that when w runs over all computable sequences of rationals, $\{W\}$ is an effective enumeration of all c.e. sets. It is now not very hard to see that the satisfaction of (3.4) $\mathcal{N}_w, \mathcal{M}_w$ implies

$$A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x \Rightarrow A_w \leq_{\mathrm{m}} A_z \ \vee \ A_x \leq_{\mathrm{m}} A_w \tag{3.5}$$

which is what we want. Indeed, if we suppose $A_z \leq_{\mathrm{wtt}} A_w \leq_{\mathrm{wtt}} A_x$ then $\mathcal{N}_w, \mathcal{M}_w$ are satisfied by their second clauses. The second clause of $\mathcal{N}_w$ implies that the disjunction in (3.4) is true. For $A_w \leq_{\mathrm{m}} A_z$, using the second clause of $\mathcal{M}_w$, we only need to decide the luck of x_i with $i \in W$ (using A_z). This is possible if the first clause of the disjunction in (3.4) is true. If not, the second clause of that disjunction gives $A_x \leq_{\mathrm{m}} A_w$.

Note that the $\leq_{\mathrm{m}}$ in (3.5) are in fact $\equiv_{\mathrm{m}}$. For (3.4) it is enough to satisfy

$$\mathcal{Q}_w : (\overline{Z} \cap \overline{A_x} \subseteq_* W) \ \vee \ (\overline{Z} \cap \overline{A_x} \cap W \text{ finite})$$

[2]consider the c.e. set $Z \cup A_x$; the maximality of Z gives $Z \cup A_x =_* Z$ or $Z \cup A_x =_* \mathbb{N}$, from which the claim follows.

$\mathcal{P}$-requirements

To guarantee the strictness of the inequality $A_z <_{\mathrm{wtt}} A_x$ we have

$$\mathcal{P} : \Phi^{A_z} \neq A_x; \phi$$

where Φ runs over the partial computable functionals. This requirement along with $\mathcal{N}$, $\mathcal{M}$ (and no other) motivate the black area.

A_z co-infinite

Finally we want x, z to be symmetric approximations (i.e. A_x, A_z infinite and co-infinite) and while $\mathcal{P}$ implies this for x, it is not obvious by what we have said so far that the same holds for z. We can easily adjust the modules described below, such that they leave infinitely many z-terms outside the black area (by restraining a finite amount of $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{M}$ action). But this is not necessary if we observe the following. Since A_x is semirecursive, it cannot be hh-simple and so it cannot be maximal. Consider a co-infinite c.e. W which contains A_x and the corresponding $\mathcal{Q}_w$. If A_z were cofinite, $\overline{A_x} \cap Z$ would be finite and thus the first clause (the hypothesis) of (3.4) would hold. By the properties of W there are infinitely many x-terms outside the black area which do not sit on w-terms. This means that the second clause of the disjunction in $\mathcal{Q}_w$ is false, and $\overline{A_x} \cap Z =_* \overline{A_x} \cap W$ must hold. But this is impossible since the first part is finite and the second infinite. So A_z will be co-infinite, provided that the requirements above are satisfied.

3.4.2 Modules

Above we showed that the requirements $\mathcal{P}, \mathcal{Q}, \mathcal{N}, \mathcal{M}$ are sufficient to imply the theorem. Before stating the strategies which will satisfy them, we say few things about the construction. As usual the black area is an increasing sequence, which we will keep implicit in this proof (e.g. expanding the black area up to a certain point means to define the current term of the sequence on that point). At the begining of stage s we define x_s between the end of the black area and the least x-term sitting outside of it. For the definition of z we have a set Z which is enumerated by various $\mathcal{Q}$-requirements and is, as before, the set of indices of x-terms which sit on z-ones. At the beginning of each stage we pick the least $n \in Z$ such that x_n doesn't sit on a z-term, and define $z_k = x_n$, where k is the least such that $z_k \uparrow$.

Hence there are *two sorts of enumerations* going on in the construction. One sort is those controlled by the black area (i.e. enumerations into A_z, A_x and the various A_w). The other is enumerations into Z and the various W. We only control (by $\mathcal{Q}$'s action) the one in Z; the one in W is done by the opponent. The two sorts of enumerations are unrelated, apart from the fact that Z-enumeration is done on the part (i.e. terms) that the the black area currently leaves unaffected.

The argument will be a tree construction, mainly because of the infinitary $\mathcal{Q}$ requirements. The black area expands according to the demand of the nodes of the tree, and at most one such expansion happens during a single stage s (and it happens in the end of it). In particular, at the end of s we let the least $\mathcal{P}, \mathcal{N}$ or $\mathcal{M}$ currently accessible node which requires attention act (note that these are finitary). But since $\mathcal{Q}$ is infinitary, we let every accessible $\mathcal{Q}$-node act (and possible enumerate into Z) at the substage of S that it is accessed.

$\mathcal{Q}$-module

This requirement is interested in

$$\overline{A_x} \cap \overline{Z} = \{b_0 < b_1 < \dots\}.$$

The strategy follows the maximal set construction, when the last is done on a tree, and so it requires nesting. Suppose that $\mathcal{Q}$ is sitting on β. The possible outcomes are $\boxed{\mathrm{i}} < \boxed{\mathrm{f}}$. Let $INF(\beta)$ be the $\mathcal{Q}$-nodes γ with $\gamma * \boxed{\mathrm{i}} \subseteq \beta$ and $FIN(\beta)$ the ones with $\gamma * \boxed{\mathrm{f}} \subseteq \beta$. The outcome $\boxed{\mathrm{i}}$ involves infinitary action and indicates

$$A_x \cap \overline{Z} \subseteq_* W_\beta$$

(where W_β is the c.e. set associated with the β's requirement); and $\boxed{\mathrm{f}}$ indicates

$$(A_x \cap \overline{Z}) \cap W_\beta \text{ finite.}$$

The module enumerates elements of $\overline{A_x} \cap \overline{Z}$ that have not appeared in W_β, into Z thus trying to make almost all b_n elements of W_β. But it acts only in expansionary stages which indicate that there is infinite potential in W_β. The level of b_n below which the work has already been done is

$$\ell(\beta) = \min\{n \mid b_n \in A_x \cap \overline{W}_\beta \ \wedge \ n > r(\beta)\}$$

where $r(\beta)$ is a finite restraint and the values of the parameters in the expressions are, as usual, subject to the current stage. If β is on the true path we will have $\lim_s \ell(\beta)[s] = \infty$ iff $\beta * \boxed{i}$ is on the true path. The strategy is the following:

Is there $n > \ell(\beta)$ with $b_n \in \overline{A_x} \cap (\cap_{\gamma \in INF(\beta)} W_\gamma)$?

- No: do nothing
- Yes: put $b_{\ell(\beta)}, \ldots b_{n-1} \searrow Z$.

Then access $\boxed{i}$ or $\boxed{f}$ depending on whether $\ell(\beta)$ has increased *(note that if it has acted under the 'yes' clause above, it has increased).*

Finally, the restraints will guarantee that $\overline{A_x} \cap \overline{Z}$ is infinite.

$\mathcal{P}$-module

Suppose that $\mathcal{P}$ is sitting on β. The point here is that we need to impose suitable restraints, as we want each b_n to reach a final value. And indeed, b_n can be agitated either by a Z-enumeration or by an expansion of the black area (and so by P's action). Moreover we want a witness for P that is not a z-term (otherwise its enumeration may interfere with the use of the computation we want to preserve). We have two restraints; r for Z-enumeration and q for the expansion of the black area. In some of the strategies, r and q restraints are imposed by saying 'we r-restrain ...' etc. Let $s(\beta)$ (at a given stage) be the largest number that the nodes to the left of β have mentioned so far. Then *we define* $r(\beta)$ to be the least number greater than $|\beta|$, $s(\beta)$ and the numbers that are currently r-restrained by the nodes above β.

We also define $q(\beta)$ to be the least of $x_{b_{|\beta|}}$, $x_{s(\beta)}$ and the (rational) numbers that are currently q-restrained by the nodes above β. This restraint requires $(q(\beta), 1)$ to stay outside the black area. Note that some x_n which contribute to q may be currently undefined. In this case we use the convention that every $x_i \downarrow$ outside the black area with $i < n$ is q-restrained (this is reasonable since undefined terms will be later defined outside the black area). Note that $r(\beta), q(\beta)$ are the restraints that β should respect. The $\mathcal{P}$-strategy is the following:

1. Pick a witness $n > r(\beta)$ with $n < \ell(\gamma)$ for all $\gamma \in INF(\beta)$ and x_{b_n} is not $q(\beta)$-restrained. Now r-restrain b_n and q-restrain x_{b_n}. *The requirement $n < \ell(\gamma)$ ensures that no higher node will put $b_n \searrow Z$. And for the lower nodes this is forbidden by The r-restraint we impose. Note that b_n will keep the current value until we reach step 4.*

2. Wait until
$$\Phi^{A_z}(b_n) \downarrow= 0; \phi. \tag{3.6}$$
Output $\boxed{\mathrm{w}}$. *If (3.6) never happens, x_{b_n} will stay outside the black area and the disagreement will witness the satisfaction of $\mathcal{P}$.*

3. Expand the black area up to the maximum z-term in the use of (3.6), less than x_{b_n}; and wait until (3.6) is restored. Output $\boxed{\mathrm{p}}$. *Because of the choice of n, this action respects the restraints of higher priority nodes. If (3.6) is never restored we win as before.*

4. Expand the black area up to x_{b_n} and q-restrain the least z-term below the use, not in the black area. Output $\boxed{\mathrm{d}}$. *We have created a disagreement which will be preserved due to the restraints we impose.*

$\mathcal{N}$-module

In $\mathcal{P}$-strategy we where able to pick a suitable witness and, by imposing restraints, keep it suitable until we diagonalise. In the $\mathcal{N}$-strategy we describe below we don't have this ability. We can try and find a suitable witness, but anytime after that, it may become unsuitable and so we have to change it. This situation may occur infinitely often and give us a useful infinitary outcome. The key idea is not to impose any restraint during these cycles. If $\mathcal{N}$ is attached to β, the strategy is the following:

1. Pick the least $n \in Z$ with $x_n \downarrow< q(\beta)$ and $n \notin W \cup A_x$.

2. Wait until one of the following happens:

   - $n \searrow W \cup A_x$
   - $\Phi^{A_w}(n) \downarrow= 0; \phi$

   Output $\boxed{\mathrm{w}}$.

3. If the first clause holds, go to step 1; otherwise proceed to the next step. *If the first clause fails and we get the computation, x_n will not be sitting on a w-term below the use (otherwise we would already have found this out and returned to step 1).*

4. Restrain x_n with q. Expand the black area up to the maximum $w_i < x_n$ with $i < \phi(n)$ and wait until $\Phi^{A_w}(n) \downarrow= 0; \phi$ is restored. Output $\boxed{\mathrm{p}}$. *If the computation*

is not restored we are done; otherwise the use will be the same, and so x_n will continue to be different than all w-terms below the use.

5. Expand the black area up to x_n and q-restrain the least w-term below the use, which lies outside the black area. Output $\boxed{\text{d}}$. *The disagreement will be preserved because of the remark in the previous step.*

If the module visits infinitely often steps 1,2,3, $\mathcal{N}$ is satisfied by its second clause and the outcome is $\boxed{\text{w}}$. If it gets stuck in step 2, it is satisfied by its first clause and the outcome is again w. If we get to 3 or 4 we are able to keep a suitable witness and so it is satisfied by its second clause and the outcome is $\boxed{\text{p}}$ or $\boxed{\text{d}}$ respectively.

$\mathcal{M}$-module

This is similar to the one for $\mathcal{N}$.

1. Pick the least n with $w_n < q(\beta)$ and $n \notin A_w \cup X_w$.

2. Wait until one of the following happens:

   - $n \searrow A_w \cup X_w$
   - $\Phi^{A_x}(n) \downarrow = 0; \phi$

   Output $\boxed{\text{w}}$.

3. If the first clause holds, go to step 1; otherwise proceed to the next step. *If the first clause fails and we get the computation, w_n will not be sitting on an x-term below the use (otherwise we would already have found this out and returned to step 1).*

4. Restrain w_n by q; expand the black area up to the maximum x-term below the use and smaller than w_n. Wait until $\Phi^{A_x}(n) \downarrow = 0; \phi$ is restored. Output $\boxed{\text{p}}$.

5. Expand the black area up to w_n and q-restrain the least x-term below the use and $> w_n$. Output $\boxed{\text{d}}$.

3.4.3 Construction

Before stating the construction we give a brief account of the restraints we impose. The r-restraint is only taken into account by $\mathcal{Q}$ and $\mathcal{P}$ nodes; and only $\mathcal{P}$-nodes contribute to it (in a $\boxed{\mathrm{w}}$-outcome). The q-restraint is taken into account by $\mathcal{N}$, $\mathcal{M}$, $\mathcal{P}$. And in fact these are the only modules that contribute to it.

We agree on a uniform labelling of the tree which is made out of the outcomes we defined in the modules above. Let this labelling be based on the following priority list

$$Q_0 > N_0 > M_0 > P_0 > Q_1 > \dots$$

The *construction* proceeds in stages, by accessing a branch of nodes of length s at stage s, according to their current outcome. While accessing the branch, we only execute the modules of the $\mathcal{Q}$-nodes or the $\mathcal{N}$ or $\mathcal{M}$ nodes that are in their first or second step (i.e. their infinitary part). For the other nodes we follow their last outcome. In the end of the stage we run the module of the highest accessible $\mathcal{P}$, $\mathcal{N}$ or $\mathcal{M}$ node that requires attention. These modules require attention when they are in a wait-type outcome (i.e. $\boxed{\mathrm{w}}$, $\boxed{\mathrm{p}}$) and they are ready to move on to the next step.

3.4.4 Verification

Obviously there is an infinite leftmost infinitely often visited path f. By induction we show that it is the true path, i.e. that every node on it (and its parameters) under any final (i.e. sitting on the true path) outcome behaves as described in the analysis of outcomes above, and so the requirement attached to it is satisfied. In other words that it satisfies the *working hypothesis* as we formulate it below.

- $\overline{A_x} \cap \overline{Z}$ *is infinite.*
- *For every* $\beta \subset f$, $r(\beta), q(\beta)$ *come to a limit.*
- *If* β *is a* $\mathcal{Q}$*-node and* $\beta * \boxed{i} \subset f$ *then* $\lim_s \ell(\beta)[s] = \infty$ *and* $|\overline{A_x} \cap \overline{Z} \cap W_\beta| = \infty$.
- *If* β *is a* $\mathcal{Q}$*-node and* $\beta * \boxed{f} \subset f$ *then* $\lim_s \ell(\beta)[s] < \infty$ *and* $|\overline{A_x} \cap \overline{Z} \cap W_\beta| < \infty$.

We can show straightaway that $\overline{A_x} \cap \overline{Z}$ is infinite. Indeed, if not there would be a least n such that $\lim_s b_n^s = \infty$. Because of the restraints r, q, after some stage no node to the right of f will be allowed to change the value of b_n. the same holds for the nodes to the left of f, because they are accessed only finitely many times. And again because of the restraints, only the nodes in $f \restriction n$ (i.e. those of length $< n$) can agitate b_n. By finite induction it is easy to see that every $\mathcal{P}$, $\mathcal{N}$, $\mathcal{M}$ node in $f \restriction n$ acts (i.e. expands the black area) only finitely often. So they stop agitating b_n and there must be a $\mathcal{Q}$-node β of maximal length in $f \restriction n$ that enumerates the value of b_n into Z

infinitely often. But this cannot be: when it does it again (after the nodes mentioned above have stopped agitating b_n and no node $<_L f \upharpoonright n$ becomes accessible) it will give b_n a value in $(\cap_{\gamma \in INF(\beta)} W_\gamma) \cap W_\beta$ and according to the module of $\mathcal{Q}$, none of the nodes $\subseteq \beta$ will enumerate the current value of b_n into Z (the ones with $\boxed{\text{f}}$-edges because they have stopped acting). So, since β was chosen maximal, b_n will not change again, a contradiction.

Suppose that $\beta \subset f$ and the working hypothesis holds for all $\gamma \subset \beta$; also that the corresponding requirements are satisfied. We show the same for β. Let us be in a final segment of stages such that no node $<_L \beta$ becomes accessible and $r(\beta)$, $q(\beta)$ have reached their final values.

β in $\mathcal{Q}$ case

First suppose that β is a $\mathcal{Q}$-node. We show that

- If $\beta * \boxed{\text{i}} \subset f$ then $|\overline{A_x} \cap \overline{Z} \cap W_\beta| = \infty$
- If $\beta * \boxed{\text{f}} \subset f$ then $|\overline{A_x} \cap \overline{Z} \cap W_\beta| < \infty$.

If the first clause didn't hold, there would be a least $n > r(\beta)$ such that $b_n \notin W_\beta$. After b_n takes its final value, β ceases to act (because any action would agitate b_n) which is a contradiction (since $\boxed{\text{i}}$ implies infinite action). Moreover $\lim_s \ell(\beta)[s] = \infty$ because otherwise there would be a $b_n \notin W_\beta$ with $n > r(\beta)$; and this cannot hold since, given that $|\overline{A_x} \cap \overline{Z} \cap W_\beta| = \infty$, β would change it to a value in W_β. In particular we get that $\overline{A_x} \cap \overline{Z} \subseteq_* W_\beta$ and so $\mathcal{Q}$ is satisfied.

The second clause is obvious if we consider the module of $\mathcal{Q}$. And of course $\lim_s \ell(\beta)[s] < \infty$ is also easy to see.

β in $\mathcal{P}$ case

Suppose that β is a $\mathcal{P}$-node. From what is said in the induction hypothesis about the $\gamma \in INF(\beta)$ it follows that β will find a suitable witness b_n. Now as long as we wait for $\Phi^{A_z}(b_n) \downarrow = 0; \phi$, b_n will not enter $A_x \cup Z$ (and nor do any b_t, $t < n$) because of r, q and the fact that the $\mathcal{N}, \mathcal{M}, \mathcal{P}$ nodes above have ceased to act. If we get stuck on step 2 (i.e. $\boxed{\text{w}}$) we are done. Otherwise we proceed to 3 and if we get stuck there (on $\boxed{\text{p}}$) we are done; if not, we end up in 4 where the computation is preserved due to q, and so we are done (on a $\boxed{\text{d}}$-outcome). It is easy to see that the restraints come to a limit.

β in $\mathcal{N}$ or $\mathcal{M}$ case

Suppose that β is an $\mathcal{N}$-node. If we never escape steps 1,2,3 we get $Z \cap \overline{A_x} \subseteq_* W$, a stable outcome $\boxed{\text{w}}$ and no restraints. If we manage to go to 4 but no further, we get a stable $\boxed{\text{p}}$ and a finite q-restraint. And if we make it to 4 we get a final $\boxed{\text{d}}$ and a finite q-restraint. The analysis for β an $\mathcal{N}$-node is similar.

To finish the proof we show that the sequence x converges. We know that $\overline{A_x}$ is infinite, and so that there exists an increasing sequence (n_i) such that the sequence (x_{n_i}) lie outside the black area. Also, according to the way we define the terms, (x_{n_i}) is decreasing, bounded and so it converges. The black area also converges at some y and, as in the proof of (13), it is enough to show that the two limits coincide. Define

$$\begin{aligned} a_1 &= x_{n_1} \\ a_{s+1} &= y + \frac{a_s - y}{2} \end{aligned}$$

It is straightforward that $\lim_s a_s = y$. If we prove that

$$a_s \geq x_{n_s}$$

for all s, using the fact that $x_{n_s} \geq y$ for all s, we get that $\lim_s x_{n_s} = y$ and finish. We prove it inductively: for $s = 1$ it is evident. Suppose that it holds for s. Let y_s be the right end of the black area at stage s. Note that when $x_{n_{s+1}}$ is defined, x_{n_s} is already defined and so

$$x_{n_{s+1}} = y_t + \frac{y_t - x_k}{2}$$

for some t, k, with $x_k \leq x_{n_s}$ and $y_t \leq x$. By the induction hypothesis, we also have $x_k \leq a_s$, and so

$$x_{n_{s+1}} \leq y + \frac{y - a_s}{2} = a_{s+1}$$

and we are done.

Chapter 4

Approximation Representations for Reals and their wtt-degrees

4.1 Introduction

We are interested in Δ_2 reals and in particular on their effective approximation properties. By a well known fact, these are the reals that are limits of computable sequences of rationals. To study these properties we introduced (see chapters 2, 3) a notion of an *approximation representation* of a Δ_2 real. Let $x = \lim z$ where z is a computable sequence of rationals converging symmetrically to (i.e. having infinitely many terms on each side of) x. These assumptions will be made without notice throughout this chapter. We say that the set

$$A_t = \{i \mid z_i < x\}$$

is the *approximation representation* (or simply representation) of x, corresponding to z. Obviously, a real can have more than one representation. The set A_t represents the way that z approximates x. Note that we are not studying the left cuts of Δ_2 reals, but the set of indices of the terms of some z converging to x, which are on the left of x. So the work in Calude et al. [9] is different from ours. We have shown a number of results about approximation representations in chapters 2, 3 which we do not need to repeat here. So detail relating to any facts mentioned below which are not entirely obvious can be found in the relevant pages.

So far we have been especially interested in c.e. representations. A representation of a real is c.e. iff the real is c.e. (in the sense that there is a computable increasing sequence of rationals converging to it). So in the rest of this chapter we assume that all reals and representations are c.e. There are three main questions we would like to ask:

- First, how do the representations of a fixed real x relate computationally to each other? We have shown that they are all T-equivalent but with respect to stronger reducibilities like wtt or m, they may (or may not, depending on the real) be very varied. Also, under a strong reducibility r they form a substructure of the r-degrees inside the T-degree of x.

- Secondly, given some representation, how do the reals which have this representation relate to each other computationally? Of course they are T-equivalent, but we show in the following that they can be quite diverse with respect to stronger reducibilities. Theorem 15 says that there is a wtt-computably independent set of c.e. reals which have a common representation A. A (countable) set of reals $\{x_i\}$ is wtt-computably independent if $x_i \not\leq_{\mathrm{wtt}} \oplus_{j \neq i} x_j$ for all i. This means that no element x_i can be computed in a wtt fashion from the rest of the elements in the set.[1]

- The third goal we want to achieve is a complete characterization of the (c.e. of course) wtt degrees which contain representations (with reference to any real). In chapter 3 we characterized representations as the *c.e. cuts of computable orderings of order type* $\omega + \omega^*$ (here ω^* is the inverse of ω). So these sets are quite interesting in many ways, and it is natural to ask which degrees they occur in. They obviously occur in every T-degree, and so we turn to look at stronger reducibilities (such as wtt). We have shown in chapter 2 that any non-computable representation (which is what we are really interested in) is a hypersimple set; and since the wtt-complete degree contains no hypersimple sets (by a classical result), it contains no representations. So indeed there are representation-free c.e. wtt-degrees. But are there hypersimple such degrees? In theorem 16 we show not only that there are, but also that there is a certain freedom in constructing them. In fact we construct entire upper cones of wtt-degrees, free of representations. By an upper cone (with bottom $\mathbf{a}$) we mean the set $\{\mathbf{x} \mid \mathbf{a} \leq \mathbf{x}\}$ (for a fixed notion of degree, and the order associated with it). The proof and particularly the strategy for the cone construction is especially interesting, as we have not encountered it before. In theorem 17 we show downward density of the representation wtt-degrees (i.e.

[1] this is analogous to the term 'recursively (or computably) independent' which refers to T-reducibility.

the ones containing representations) in the c.e. ones. In other words, any non-computable c.e. set wtt-bounds a non-computable representation.

In theorem 18 we construct a non-zero T-degree which bounds no bottom of a representation-free cone of wtt degrees (like the ones constructed in theorem 16). The proof of this result is especially interesting as it is an infinite injury where the restraint imposed by a single requirement can tend to infinity (i.e. has no $\liminf$).

Unexplained notation in this chapter is quite standard. When we write $\Phi^A = B; \phi$ we mean that the reduction of B to A is wtt (i.e. that the use ϕ is computable).

In priority constructions (particularly) it is very helpful to have an intuitive picture of what's going on. For this reason we describe briefly how we picture the construction of a representation A. We define a sequence z (which will eventually tend symmetrically to a limit) and a non-decreasing sequence y in $[0, 1]$. Our aim is to build A_z $(= A)$ so that it satisfies certain computational properties. Whenever we want to enumerate a number $n \searrow A_z$ we wait until $z_n \downarrow$ and let y be greater than z_n. The interval $[0, y_s]$ is called the *black area* (at stage s) and so enumeration into A_z is done by expansion of the black area. The distinctive feature of representation constructions is that when you enumerate $n \searrow A_z$ you *have to* enumerate all k such that $z_k \leq z_n$ into A_z. An illustration is given in figure 4.1.

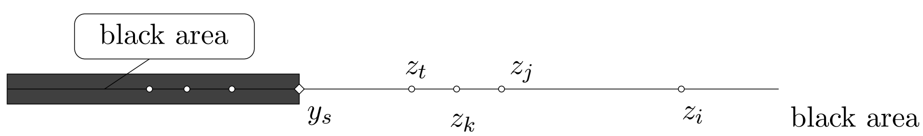

Figure 4.1: Representation constructions

The results in this chapter are published in [6].

4.2 Different reals with common representation

We begin with

Theorem 15. *There is a wtt-computably independent set of c.e. reals which have a common representation A.*

Proof. We want to build a representation A and symmetrically converging sequences $z^i \to x_i$ such that $A_{z^i} = A$ and $x_i \not\leq_{\text{wtt}} \oplus_{j \neq i} x_j$ for all $i \in \mathbb{N}$. Our requirements are

$$N_{\langle e,i\rangle} : \Phi_e^{\oplus_{j\neq i}x_j} \neq x_i; \phi_e$$

and we are going to build the sequences and reals in our usual framework. For each real x_i we have a sequence y^i which converges monotonically to x_i. At any stage s the interval $[0, y^i_s]$ is the *i-black area* and y^i_s is our current approximation for x_i. At all times we ensure that all A_{z^i} are equal to the representation A we are constructing. This means that if the i-black area expands and covers new z^i-terms, we *assume* that the indices of these terms are enumerated into A. Moreover we motivate the expansion of other j-black areas (i.e. for those j for which there are *defined* z^j terms outside the j-black area) so that we preserve $A = A_{z^j}$ for all j. This *chain reaction* will happen for only finitely many j since at any given stage only finitely many z^j terms (for any j) are defined. In fact, at stage s we define z^j_s for all $j \leq s$ (so at s the defined z-terms are z^j_t for all $j \leq t \leq s$).

All the parameters in the construction will be finite binary rationals (i.e. rational numbers with a finite binary expansion). The strategy to satisfy $N_{\langle e,i\rangle}$ is the following: we start at a stage s by choosing a finite binary sequence q such that $q0 \sqsubseteq y^i_s$ (we think of rationals both as binary expansions and binary sequences). This can be done at any stage since y^i_s is finite and can be assumed to have a suffix of any (finite) number of zeros. We impose restraints (on the growth of y^i) to ensure $q0 \sqsubseteq x_i$ and wait until $\Phi_e^{\oplus_{j\neq i}x_j}(n) \downarrow = 0; \phi_e$ where $n = |q| + 1$. If we never get this computation, our restraints will guarantee the satisfaction of $N_{\langle e,i\rangle}$.

If we get it, say at stage s_0, we would like to define $y^i = q1$ (in order to create a disagreement). But this increase in y^i may motivate an enumeration into A and so (by the chain reaction described above) a change in $\oplus_{j\neq i}x_j$ below the use. In this case we will not be able to preserve the disagreement. To deal with this problem, we first set y^i to be the largest z^i-term less than $q1$ and j-restrain $(p_j, 1)$ where

$$p_j = y^j \upharpoonright \phi_e(n) + 2^{-s_j} \tag{4.1}$$

and s_j is the largest 0-position in $y^j \upharpoonright \phi_e(n)$, for all j involved in the use (for those j that $s_j \uparrow$ we do not put any restrain). We also require any new z^j term to be defined outside $(p_j, 1)$, for those j (so that a following action $y^i = q1$ will only cause changes in the expansion of x_j outside the use). Now we wait until $\Phi_e^{\oplus_{j\neq i}x_j}(n) \downarrow = 0; \phi_e$. If we don't get it, $N_{\langle e,i\rangle}$ is satisfied as before. Otherwise the use will be the same and setting $y^i = q1$ will create disagreement without spoiling the computation. That is because if x_j changed below the use, this would be because some term z^j_t motivated a

y^j expansion (due to a $t \searrow A$ enumeration). But such z^j_t terms were defined after stage s_0, and so were defined in order not to motivate any such change in the expansion of x_j below the use (which was the same as now). Hence this would lead to a contradiction.

Finally we will define z^i_s in the middle of (y^i_s, m') where $m' = \min\{m, z^j_k \mid k \notin A[s]\}$ and $(m, 1)$ is the strictest j-restraint imposed currently. This ensures that the z^j-terms are defined close enough to $\lim y^j$ so that $\lim z^j = \lim y^j$. Next, we lay out the formal module for $N_{\langle e,i\rangle}$ which is actually a part of the construction. Note that here we take into account restraints imposed by higher priority requirements. During the construction each requirement imposes j-restraints for various j. Such a restraint is of the form 'don't let y^j enter $(p, 1)$'. Note these restraints imply restraints on A: if $(p, 1)$ is j-restrained and contains z^j_k then k is restrained from A.

1. Choose a prefix of the current y^i-approximation to x_i with last digit 0, i.e. some q with $q0 \sqsubseteq y^i[s]$) such that $q1$ is not i-restrained by a higher requirement, and it doesn't sit on any defined z^j-term for any j. i-restrain $(q1, 1)$.

2. Wait until

$$\Phi_e^{\oplus_{j \neq i} x_j}(n) \downarrow = 0; \phi_e \tag{4.2}$$

   where $n = |q| + 1$.

3. Let $y^i = z^i_t$ where z^i_t is the largest z^i-term less than $q1$. For each j involved in the use of (4.2) j-restrain $(p_j, 1)$ where p_j is defined in (4.1). Wait until (4.2) is restored. *By this action, A-enumeration occurs and so, various y^j-black areas move. This will not affect higher priority requirements because $(0, q1)$ is not restrained by them.*

4. Drop the i restraints of step (1) and set $y^i = q1$; also i-restrain $(z^i_k, 1)$ where z^i_k is the least z^i-term $> q1$. *The use $\oplus_{j \neq i} x_j \upharpoonright \phi_e(n)$ doesn't change because for the $k \searrow A$ by this action, z^j_k were defined after step (3) and so they are $< p_j$ (by the way we define z-terms, see below). The disagreement will be preserved by keeping the i-restraints of this step and the j-restraints of step (3).*

Now the *construction* is as follows. For all j set $y^j_0 = 0$. At $s > 0$

(a) Define z^j_s (for each $j < s$) in the middle of (y^j, m') where $m' = \min\{m, z^j_k \mid k \notin A\}$ and $(m, 1)$ is the strictest j-restraint imposed by any requirement at the moment.

(b) Let the least requirement which requires attention (i.e. is ready to move step) act and initialize lower requirements (i.e. set their modules in step (1) and cancel their restraints).

(c) For all j, k, if $y^j_{s-1} < z^j_k$ and $k \in A$ then set $y^j_s = z^j_k$ for the max such z^j_k. *This ensures* $A = A_{z^j}$ *for all* j.

Now we do the *verification* of the construction. It is evident that for all $n, k \notin A$,

$$n > k \iff z^j_n < z^j_k$$

for all j. And so, by step (c) of the construction, $A = A_{z^j}$ for all j. Now we prove by induction that each $N_{\langle e,i\rangle}$ is satisfied and eventually ceases requiring attention. The induction step for $N_{\langle e,i\rangle}$: assume that after s_0 no higher priority requirement requires attention. $N_{\langle e,i\rangle}$ will receive attention and step (1) of the module will be performed. Note that y^j, z^j, p_j, q all take values of finite binary rationals $\mathbb{Q}_2$ since this set is closed under addition and division by 2. So, since only finitely many restraints are imposed by higher priority requirements, q will be found in step (1). If we wait forever in step (2) of the module, we are done since then $x_i < q1$ ($y^i \not\to q1$ because of the infinitely many requirements with empty functionals, and the restraint they impose in (1)).

Otherwise we pass on to (3) and, if stuck forever, we are done for the same reasons. Otherwise the use of the restored computation is again $\phi_e(n)$ and $\oplus_{j\neq i} x_j \restriction \phi_e(n)$ same as just after we acted in (3), due to the j-restraints and the induction hypothesis. So we pass on to (4) and the z^i-terms in $(y^i, q1)$ have indices $k > s_1$, the stage when (3) was executed. So for those k and the j involved in the use of (4.2), $z^j_k < p_j$ and so, enumeration into A will not spoil (4.2) (under step (c) of the construction). So at step (4) we put $y^i = q1$ and preserve the disagreement by restraining $(z^i_k, 1)$.

This concludes the induction step and the only thing left to show is that $\lim y^i = \lim z^i$ for all i. Fix i: y^i converges as non-decreasing and bounded. By the construction we have

$$z^i_{s+1} \leq \frac{y^i_{s+1} + \lambda^i_s}{2} \tag{4.3}$$

where $\lambda^i_s = \min\{z^i_k \mid k \notin A[s]\}$. Let $\{j_s\}$ be a monotone enumeration of $\mathbb{N} - A$. By (4.3),

$$z^i_{j_{s+1}} \leq \frac{y^i_{j_{s+1}} + z^i_{j_s}}{2}.$$

Let

$$
\begin{aligned}
a_0 &= z^i_{j_0} \\
a_{s+1} &= \frac{x_i + a_s}{2}.
\end{aligned}
$$

For all s, $a_s \geq z^i_{j_s}$; indeed, it holds for $s = 0$ and if $a_s \geq z^i_{j_s}$ then

$$a_{s+1} = \frac{x_i + a_s}{2} \geq \frac{y^i_{j_{s+1}} + z^i_{j_s}}{2} \geq z^i_{j_{s+1}}.$$

So $\lim_s a_s = \lim_s z^i_{j_s} = x_i$. Now it is easy to see that $\lim_s z^i_s = x_i$, which finishes the proof. $\square$

4.3 Wtt-degrees of representations

We noticed in chapters 2, 3 that any representation of a non-computable c.e. real is a hypersimple set. And since the wtt-complete degree contains no hypersimple sets (by a well known result), this degree contains no representations. This raises the question which c.e. wtt degrees contain representations (note that every c.e. T-degree contains such). Are they the hypersimple ones? The following theorem says that there are entire upper cones of wtt-degrees, free of representations. Moreover the bottoms of these cones can avoid any specified non-trivial upper cone of T-degrees; and can even have hypersimple wtt-degree (which means that the wtt-degrees containing representations are properly contained in the hypersimple ones).

In chapter 3 we also noted that the representations of c.e. reals (from now on, just *representations*) are exactly the c.e. cuts of computable orderings of $\mathbb{N}$ of order type $\omega + \omega^*$. So the results below can be stated in terms of computable orderings. If A is a representation, there is a computable ordering of $\mathbb{N}$ determined by a computable function ψ (i.e. $n \prec m \iff \psi(n, m) = 1$) whose (say) left cut is A. Then we say that *A is a representation via ψ*. Let $\{D_n\}$ be an effective sequence of all finite sets.

Theorem 16. *Let C be a non-computable c.e. set. There is $A \not\geq_T C$ hypersimple such that for all c.e. $W \geq_{wtt} A$, W is not a representation.*

For theorem 16 we need to satisfy the following requirements:

$$\begin{aligned}
\mathcal{N}_{\Phi,W,\psi}: &\quad \Phi^W = A; \phi \Rightarrow W \text{ not a representation via } \psi \\
\mathcal{H}_{\varphi}: &\quad \exists n (D_{\varphi(n)} \cap A = \emptyset) \\
\mathcal{C}_{\Phi}: &\quad \Phi^A \neq C
\end{aligned}$$

where Φ, ϕ, ψ run over the computable functionals/functions, and W over the c.e. sets. The strategies for $\mathcal{C}, \mathcal{H}$ (guaranteeing cone-avoidance in the Turing degrees and hypersimplicity) are well known, but we will state them briefly. A new strategy is described for $\mathcal{N}$. Roughly, to satisfy $\mathcal{N}$ we will start enumerating $\overline{W}$ (via an auxiliary set D) using the hypothesis that W is a representation via ψ (and of course $\Phi^W = A; \phi$). If at some point our guess D for $\overline{W}$ fails (i.e. an element of D appears in W) then we will be able to satisfy $\Phi^W \neq A; \phi$ by creating and preserving a disagreement.

Let us discuss this plan in more detail. We are given W and ψ, and we may assume that W *is* a representation via ψ in order to try to destroy $\Phi^W = A$. If this hypothesis fails, $\mathcal{N}_{\Phi,W,\psi}$ is satisfied. So we may think W, ψ as the construction of a sequence of rationals converging symmetrically to a real, which produces the representation W of that real. In terms of our framework, the black area is controlled by the enumeration in W and the relative position of the terms of the sequence is determined by ψ (this description gives us a picture of what we are trying to control, i.e. the procedures given to us by the opponent).

The strategy consists of two recursive procedures A, B. The first one consists of potentially infinitely many cycles A_n, each of which builds upon the work done on its predecessor A_{n-1}. The purpose of A_n is to find and enumerate elements to D (so that we are closer to $D = \overline{W}$). Suppose that W is a representation via ψ. The main idea behind D-enumeration is that any $d \in \overline{W}$ has only finitely many ψ-successors. Now, considering a d which is apparently in $\overline{W}$ (i.e. has not yet been enumerated in W) we look for a set I of witnesses (intended for Φ-diagonalizations) such that the set R of their rectification codes (i.e. numbers currently outside W and below the use of at least one Φ-computation on an argument in I) which are ψ-greater than d is smaller (in cardinality) than I itself. Since W is a representation via ψ, there are only finitely many elements ψ-greater than d and so such a set I will be found provided that d is indeed member of $\overline{W}$.

Once we find I (and d is still outside W) we have reasons to believe that d is not going to appear in W later on, and we enumerate it in D. Our belief comes from the following fact: if later on $d \searrow W$, every element not ψ-greater than d will enter W (since the later is a representation via ψ) and so we hold a set I of witnesses whose overall rectification codes are less than their actual number. This means that can we start a diagonalization ripple which ensures a final $\Phi^W \neq A; \phi$ disagreement: for each I-diagonalization at least one element of R will enter W to rectify it and so there will be a final I-diagonalization which is impossible to rectify. The diagonalization procedures is the content of steps B_n.

Of course there is the possibility that during the process of searching for I, d is enumerated in W. In this case we have to pick a different d. Since $\overline{W}$ is infinite, we will eventually come up with a suitable d. Moreover when we enumerate $d \searrow D$ we can enumerate all numbers ψ-greater than d as well (since if any of these appear in W, d must also appear). This feature, along with choosing d as ψ-small as possible (see parts (a),(b),(c) of step 2 of A_n below) ensures that if this procedure is not interrupted (e.g. by $D \cap W \neq \emptyset$), it will give the whole $\overline{W}$.

So if indeed the hypotheses of $\mathcal{N}$ hold and W is a representation via ψ, $D \cap W = \emptyset$ and according to the above, $D = \overline{W}$. So W is computable. In other words, we have satisfied the requirement:

$$\mathcal{N}'_{\Phi,W,\psi} : \Phi^W = A; \phi \Rightarrow W \text{ not a representation via } \psi \text{ or } W \text{ is computable.}$$

In fact, we can let all strategies like the one described (i.e. for all Φ, W, ψ) work together without any interference. Indeed, each strategy chooses witnesses from a special set (disjoint from the sets of other strategies) and so there is no injury (the only restraints set by the strategy are on witnesses). What we achieve is the satisfaction of all $\mathcal{N}'$. But this obviously implies that A is non-computable. Using this fact it is now clear that the satisfaction of $\mathcal{N}'$ implies the satisfaction of $\mathcal{N}$ (since the computability of W and $\Phi^W = A; \phi$ implies the computability of A).

This is an interesting phenomenon: $\mathcal{N}'$ can be regarded as pseudo- requirements which are individually weaker than the main requirements and whose satisfaction is the direct outcome of our strategy. However the satisfaction of all of them (which is the direct outcome of our construction) implies the satisfaction of all of the real requirements. The 'outcome' W *is computable* can be regarded as a pseudo-outcome of $\mathcal{N}'$ since it is never the outcome of a strategy in the sense that no strategy will end up with D infinite (and so, $D = \overline{W}$ according to the above analysis). This is an implication of the fact that A ends up incomputable (and so $\Phi^W \neq A; \phi$, if W is computable, which means that we only get finitely many expansionary stages and

D finite). What happens here is that the D-enumeration is a pseudo-strategy which always fails, but it pushes the satisfaction of the pseudo-requirements in different ways (diagonalization and representation property failure).

As a byproduct of this analysis we get that no strategy is going to run for ever. Each family of steps $A_0, A_1, \ldots$ must stop in a final A_n (and of course the family $B_0, B_1, \ldots$ does not have the potential of running for ever, see below). So, an $\mathcal{N}'$ strategy (like the one discussed above and described below) runs finitely often thus imposing only a finite restraint on numbers of its special set U. This feature allows us to add the hypersimplicity requirements $\mathcal{H}$. These strategies will always respect the higher priority $\mathcal{N}'$ strategies and when they act they will initialize the lower priority strategies. Finally the only effect that the cone avoidance strategies $\mathcal{C}$ have in the strategies discussed above is a Sacks restraint with $\liminf < \infty$ as in the usual Sacks argument done on a tree. So if we transfer our $\mathcal{N}', \mathcal{H}$ strategies on the usual tree that is used in the cone-avoidance strategy the whole construction works without any special modifications. We now formally state the strategy for $\mathcal{N}'$.

Let $n = 0$, $D = \emptyset$. We assume all functional uses increasing, and a fixed restraint r that the strategy is asked to respect by higher priority strategies (in the complete tree construction this will be the $\liminf$ of one or more Sacks restraints lying on the tree above the strategy and a fixed restraint from the nodes on the left). As mentioned above, each $\mathcal{N}'$ strategy chooses A-witnesses from a special set U disjoint from the sets of other $\mathcal{N}'$ strategies. This very strategy imposes its own restraint but this is only on numbers of its special use set U and so they only affect lower $\mathcal{H}$ requirements. The module for $\mathcal{N}'_{\Phi,W,\psi}$ is as follows (the various parameters like n, D may be reassigned values after running the module):

A_n (D-enumeration step)

1. If $D \cap W \neq \emptyset$ then wait until some $d_k \searrow W$ and go to step B_k. *In order to start A_n we must ensure that the previous A_i steps look successful, i.e. $D \cap W = \emptyset$. If they do we proceed to the main clauses of A_n; otherwise we wait until some $d_i \searrow W$; these elements d_i control the D-enumerations in the sense that any element t in D must have appeared 'after' some d_i ψ-less than t was enumerated in D. So if $D \cap W \neq \emptyset$ and W is indeed a representation, some d_i must appear in D.*

2. (a) Let $\ell > 0$ be the maximum such that ψ has ordered $\mathbb{N} \upharpoonright \ell$ and wait until it takes a value greater than any previous one (including the values it

took in previous A_i steps).

(b) Choose the currently ψ-minimum element d_n in $\overline{W} \upharpoonright \ell$ and ψ-less than any number currently in D. If it doesn't exist, go to (a).

(c) Find k such that for the set I_n of the next k unused elements in U above the restraint r (i.e. the first k elements of $U - \cup_{i<n} I_i$ greater than r) the following holds: if $v_n = \max_{i \in I_n} \phi(i)$ then the number of elements less than v_n and ψ-greater than d is less than k. If during this search $d_n \searrow W$, let d'_n be the ψ-minimum element in $\overline{W} \upharpoonright \ell - D$, $d_n := d'_n$ and go to (c); if it doesn't exist, go to (a). Otherwise, if the search is complete and $d_n \notin W$ go to step 3.

The restraint r will remain the same during the life of this strategy unless it is initialized by the global construction. If ψ defines a total linear ordering of $\mathbb{N}$ of order type $\omega + \omega^*$ and W is its bi-infinite left cut, this step will be completed. Indeed, $\ell \to \infty$ (so it is impossible to be stuck on (a)) and since $\overline{W}$ has no ψ-least element, any (a)-(b)-(a) loop is only finite.

Also, no infinite loop involving (c) can occur for the following reason: any (c)-(b)-(c) loop uses a fixed ℓ and so it must be finite; so any infinite loop involving (c) must also involve (a). Now every time we visit (a), ℓ gets bigger and there will be a stage where there is an element $d' < \ell$ permanently outside W and ψ-less than any element currently enumerated in D (according to the assumptions on ψ and W). At such a stage, (b) will pick up a d_n ψ-less than or equal to d'. Now if the loop continues, (c) will have to consider d', and with this value of the parameter d_n the (c)-search cannot be interrupted. So eventually there will be a search in (c) which is not interrupted by $d_n \searrow W$. By the assumption on ψ and W such a search must terminate; indeed, d_n is permanently outside W and so it has only finitely many ψ-successors. So, as k grows all the time and "the number of elements less than v_n and ψ-greater than d_n has an upper bound, the search will finish and we will eventually pass to the next step.

Note that if any of the assumptions on ψ and W fails, the above argument does not work and we may not be able to escape this step (but this is no problem as under these circumstances $\mathcal{N}$ is satisfied).

3. Enumerate $d_n \searrow D$ and fix the values of d_n, v_n and I_n (as they were last defined above). Enumerate into D all elements less than ℓ that have been ψ-ordered greater than d_n and restrain the witnesses I_n from A. *Note that*

in the end of A_n, D only contains elements ψ-less than or equal to d_n. If we find out that some element of the current D appears in W, d_n must appear in W (or else W is not the cut we assumed it is). Upon such an event the construction will activate B_n which will start diagonalizing against $\Phi^W = A; \phi$ using I_n as the set of witnesses. Since $d_n \searrow W$, the rectification positions for any such diagonalization are less than $|I_n|$ according to (d) of step 2. So by the last diagonalization $\Phi^W = A; \phi$ will be destroyed.

4. Let $n := n + 1$ and go to step A_n.

B_k (D-failure step) We assume the values I_k, d_k, v_k as defined in step A_k.

1. Wait until $\ell(\Phi^W = A; \phi) > m$ for all $m \in I_n$ and all ψ-predecessors of d_k less than v_k enter W. *If we wait forever here, it means that W is not the left cut of the computable ordering on $\mathbb{N}$ defined by ψ, and so $\mathcal{N}$ is satisfied. Note also that d_k has less than $|I_k|$ ψ-successors less than v_k (as when it was defined).*

2. *(Diagonalization)*

   (a) Wait for a Φ-expansionary stage.

   (b) Put the least element of $I_n - A$ into A and go to (a).

   After the first diagonalization in (b), every time we leave (a) a rectification has occurred and so the set R_n of I_n-rectification codes is reduced by one. Since initially $|R_n| < |I_n|$ and for each element leaving I_n at least one element exits R_n, this (a)-(b)-(a) loop must end up in (a), unable to get a further rectification (and so, expansionary stage).

Analysis of outcomes

$\boxed{1}$ The module runs over all $A_0, A_1, \dots$ and never stops. This means that we get infinitely many Φ-expansionary stages (so $\Phi^W = A; \phi$) and $\ell \to \infty$ (so ψ defines a linear ordering on $\mathbb{N}$). It also means that $D \cap W = \emptyset$ and according to the second step of A_n, $D = \overline{W}$. So W is computable.

$\boxed{2}$ At some A_i we get stuck forever. Then either $\Phi^W \neq A; \phi$ (not giving us enough Φ-expansionary stages) or ψ does not define a linear ordering on $\mathbb{N}$ (not giving us enough ℓ-expansionary stages) or there is an infinite loop in the (a), (b), (c) clauses of step 2 of A_i. If the loop is (a)-(b)-(a) it means that $\overline{W}$ has a ψ-least element and so W is not a representation via ψ. Any other infinite loop must involve step (c) infinitely often and this means again that W is not a representation via ψ (e.g. see the comments following step 2 of A_n).

$\boxed{3}$ We end up on some B_k step. In this case $\Phi^W \neq A; \phi$ is guaranteed as we explained above.

The analysis of outcomes shows that $\mathcal{N}'$ is satisfied. The module for $\mathcal{H}_\varphi$ is to simply find a t such that $\min D_{\varphi(t)} > r$ (where r is the restraint inherited by higher priority requirements) and then empty $D_{\varphi(t)}$ into A and initialize lower priority $\mathcal{N}'$ requirements. The module for $\mathcal{C}_\Phi$ is to impose (to lower priority strategies) the restraint $r =$ the use of the computations $\Phi^A = C$ up to the first point of disagreement (or Φ being undefined).

We picture the construction on a (downwards expanding) tree. The nodes of the tree are effectively assigned strategies so that any infinite branch is equipped with strategies for each of our infinitely many requirements. An $\mathcal{N}'$ or $\mathcal{H}$ node has only one branch. A $\mathcal{C}_\Phi$ node has infinitely many branches corresponding to (and ordered as) the natural numbers. These are meant to be the various values that the restraint of this strategy takes during the construction.

During a stage s we successively access the nodes of a branch of length s, starting from the top node $\emptyset$ and going through the branch that is activated by the strategy that we have last accessed. For a $\mathcal{C}_\Phi$ node this is the branch corresponding to the current value of the restraint while for the others there is only one choice. If during a stage, an $\mathcal{H}$ strategy α enumerates into A, we initialize all lower priority $\mathcal{N}'$ strategies (so that they start anew). Lower priority strategies are the ones that are below α (i.e. their branch contains α) or to the left of it (with respect to the usual lexicographical ordering of the nodes induced by the ordering on the outcomes). Of course, when a node α becomes accessible, all strategies sitting on nodes to the left of α are initialized. The restraint r that a strategy α is asked to respect (often mentioned in the above modules) is the restraint imposed by nodes above or on the left of α.

First we verify that there is an infinite leftmost infinitely often accessible path f and $\mathcal{C}$, $\mathcal{N}'$ are satisfied. Inductively suppose that the branch $f \restriction n$ is defined (and satisfies the 'leftmost' properties). If node $f \restriction n$ is $\mathcal{H}$ or $\mathcal{N}'$ then we easily see that $f \restriction n+1$ defined by extending through the unique branch of the node, satisfies the 'leftmost' properties. If it is $\mathcal{C}$ then assuming that there is no leftmost edge infinitely often accessible we show the usual Sacks contradiction, that C is computable. So there is such edge and $f \restriction (n+1)$ is defined by adding this edge to f. This also shows that $\mathcal{C}$ is satisfied. Now that we know that f is infinite (and so it contains nodes for each $\mathcal{N}'$) we show that any $\mathcal{N}'$ strategy on f succeeds. Suppose that $\mathcal{N}'$ is not initialized anymore (such stage exists since f is leftmost and there are only finitely many $\mathcal{H}$-nodes above $\mathcal{N}'$). Then the strategy will work without any distraction (lower priority $\mathcal{H}$ requirements respect it and other $\mathcal{N}'$ requirements use different witnesses) and will deliver one of the outcomes justified in the *analysis of outcomes* above. So $\mathcal{N}'$ is satisfied.

Now, as explained above, since all $\mathcal{N}'$ are satisfied, A is non-computable. So all $\mathcal{N}$ are satisfied and also no $\mathcal{N}'$ strategy runs forever (going through $A_0, A_1, \dots$); in other words outcome $\boxed{1}$ is never realized. This means that each $\mathcal{N}'$ only imposes a finite restraint to lower priority $\mathcal{H}$ requirements, and so the later are satisfied. This completes the proof of the theorem. On the other hand we have

Theorem 17. *Every non-computable c.e. wtt-degree bounds a non-zero wtt-degree containing representations.*

To prove this theorem we combine our usual construction of a real with non-computable representation (see chapters 2, 3) with permitting. We build a sequence z which converges symmetrically to a real x, and a non-decreasing sequence y which converges monotonically to x. Let A be a non-computable set; $A_z = \{k \mid z_k < x\}$ will be our desired representation, bounded by A. We want to satisfy:

$$\mathcal{P}_\Phi : \Phi \neq A_z$$

So we carry on defining z-terms in decreasing order *outside* $[0, y_s]$ (which we often call the *black area*). When we are ready to attack some $\mathcal{P}_\Phi$ (of least priority requiring attention) we define the current term y_s up to z_k where k is the index we want to enumerate into A_z (thus expanding the black area); and so on and so forth. The observation here is that we can easily add permitting: we don't want to enumerate an index k unless some number less than k enters A at the current stage. As usual every $\mathcal{P}_\Phi$ will require attention infinitely many times unless it is satisfied. Now note that such an action for satisfying $\mathcal{P}_\Phi$ may enumerate into A_z numbers other than k (namely the indices of terms less than z_k which have not yet entered the black area). The crucial point is that all these will be greater than k (according to the way we define z) and so they will be A-permitted whenever k is so. Finally we need to keep an order on the witnesses: lower positive requirements hold larger unrealized witnesses k (i.e. with $\Phi(k)\uparrow$) and a new witness is chosen for $\mathcal{P}_\Phi$ whenever the previous one has been realized (i.e. $\Phi(k)\downarrow$). This will give a standard finite injury effect to the construction (since whenever a new witness is chosen for $\mathcal{P}_\Phi$, all lower requirements have to change theirs).

4.4 Non-bounding bottoms of representation-free wtt upper cones

The following theorem relates T and wtt computations with representations; also, its proof exhibits an interesting kind of priority.

Theorem 18. *There is a non-zero c.e. Turing degree which bounds no wtt-degree whose upper cone is free of representations.*

The requirements are

$$\begin{aligned} \mathcal{Q}_{\Phi,W}: &\quad \Phi^C = W \Rightarrow \exists A \text{ representation}(W \leq_{\text{wtt}} A) \\ \mathcal{P}_\Phi: &\quad \Phi \neq C \end{aligned}$$

and we attempt $W \leq_{\text{wtt}} A$ in $\mathcal{Q}_{\Phi,W}$ by enumerating a functional Γ with computable use γ, trying to preserve and expand the agreement $\Gamma^A = W;\gamma$.

In order to ensure that A, the set we are constructing for the sake of $\mathcal{Q}_{\Phi,W}$, is a representation, we construct a sequence z of rationals in the usual way such that $A_z = A$ (with an increasing 'black area' controlling the enumeration into A_z). By the characterization of representations as left cuts of computable orderings of type $\omega+\omega^*$ (see chapter 3) we only need to specify the position of each z-term relative to the others, when constructing z (we are not concerned with its convergence).

We define γ on numbers which are currently outside A (i.e. A_z) and make it increasing. The z-terms are defined as usual in decreasing order *outside the black area.* Now the problem is that if the black area expands up to $z_{\gamma(k)}$ (for the sake of enumerating $\gamma(k) \searrow A$) all the defined $\gamma(n)$ with $n \geq k$ will enter A. When a part of $\mathbb{N} \upharpoonright \gamma(k)$ enters A, it is not good news because our opportunities to change computation $\Gamma(k)\downarrow$ (after a possible $k \searrow W$) become fewer (as the use $\gamma(k)$ is fixed, once defined). To make things clear, we use a Γ-*marker* Γ_k for each k, which initially sits on the position (i.e. value) of $\gamma(k)$. In general, it sits on the largest number (i.e. smallest z-term) outside the black area and less than or equal to $\gamma(k)$. *The values that* Γ_k *takes are decreasing* and it could happen that eventually it has nowhere to sit (i.e. it is undefined). This is exactly what we want to avoid. We want each Γ_k to eventually rest on a number outside A (so that if k appears in W we are able to rectify the Γ-computation by enumerating the current position of Γ_k into A). Hence Γ_k being defined means that we are able to rectify Γ on k, if needed.

Now as explained above, an enumeration of some $\gamma(k)$ into A may result in the enumeration of other $\gamma(n)$ into A. This means that during the construction, many Γ-markers may occupy the same position. So if Γ_k loses its current position (to move to a smaller one) it may not be because $k \searrow W$ (but because of some other W-enumeration). So Γ_k may lose all of the positions that is allowed to have (thus ending up undefined) and still k not have appeared in W. A subsequent $k \searrow W$ will result in Γ being wrong and us being unable to rectify it.

To avoid this situation we use Φ^C to restrain W. Whenever we define $\gamma(k)$ on some number n, we make sure that the agreement $\Phi^C = W$ is higher than n and so we can restrain a subsequent movement of Γ_k (due to $k \searrow W$). More generally, whenever we place Γ_k in a new position, we make sure that we can restrain Γ_k from further movement (i.e. we wait until $\ell(\Phi^C = W)$ is big enough before enumerating the new Γ-axiom on k). Of course this strategy results $\mathcal{Q}_{\Phi,W}$ imposing a restraint r with $\liminf r = \infty$. This conflicts with the satisfaction of the $\mathcal{P}$ requirements, which can only accept a finite restraint (or at least with $\liminf < \infty$). If we were to ensure that beyond some stage, r is not violated anymore, then we would have that almost all Γ-markers never move from their initial position. We have space to be more flexible. We describe the situation of a $\mathcal{Q}_{\Phi,W}$ with highest priority and all $\mathcal{P}$ requirements (priority-ordered in some effective way) below it. After we deal with this case, the rest of the $\mathcal{Q}$ strategies can be added with only a finite injury effect (though the atomic case has infinitary nature).

We will spread out r to the lower $\mathcal{P}$ requirements. So r will be violated by lower priority requirements infinitely often, but in a nice way. In particular we define r_n (n-restraint, for $n > 0$) to be the use of $(\Phi^C = W) \upharpoonright (\ell_n + 1)$ where ℓ_n is the index of the largest Γ-marker sitting on the n-th position (i.e. number—in order of magnitude) outside the black area. If there is currently no n-th position outside the black area, or the length of agreement is less than $\ell_n + 1$, let $r_n = 0$.

Now the n-th $\mathcal{P}$-requirement below $\mathcal{Q}_{\Phi,W}$ listens to the r_m restraints for $m \leq n$; i.e. it respects $R_n = \max_{m \leq n} r_m$. To give an idea of the construction and the movement of the Γ-markers, once we state the strategy for $\mathcal{Q}_{\Phi,W}$ it will be easy to verify inductively that at any stage

$$n < m \iff \text{the } \Gamma\text{-markers on } m \text{ have bigger index than those on } n$$

(iff $z_n > z_m$) for all n, m not (yet) in A. Also it is obvious that each position permanently outside the black area, will carry at least one Γ-marker. And for any n, the markers sitting on n are protected from losing their position by restraint r_n of $\mathcal{Q}_{\Phi,W}$ (which may be violated, but only finitely often). The strategy for $\mathcal{Q}_{\Phi,W}$ is as follows:

1. (*z-definition*) Let n be the least such that $z_n \uparrow$. Define z_n outside the black area and less than any z-term outside the black area.

2. (*Γ-definition*) Let n be the least such that $\Gamma(n) \uparrow$. If $\gamma(n) \downarrow$, enumerate the axiom $\Gamma(n) = W(n)$ with use $\gamma(n) \downarrow$.

   If $\gamma(n) \uparrow$, wait until $\ell(\Phi^C, W) > n$ and there is a (largest) z_k outside the black area which carries no markers. Then, if k is the t-th (in order of magnitude) number outside the black area, define $\gamma(n) = k$ (thus putting Γ_n on z_k) and t-restrain the C-use of $\Gamma^C \upharpoonright (n+1)$ (since only Γ_n sits on z_k). *The t restraint r_t will automatically be applied according to its explicit definition.*

3. *(Γ-rectification)* Let k be the least such that

$$\Gamma^A(k) \downarrow= 0 \neq W(k)$$

   (if there is no such, do nothing). Then:

   - *Expand* the black area up to the position (say n) of marker Γ_k. *By this action we remove (temporarily) all Γ-markers with index $\geq k$ from the line. Later we will put them all on a single position; namely on the largest number $\leq n$ which is outside the black area (i.e. outside A).*
   - *Wait* until $\ell(\Phi^C, W)$ becomes larger than $\max_t(\gamma(t) \downarrow)$ (i.e. the maximum argument for which we have ever enumerated an axiom). *Before we enumerate axioms for the arguments $\geq k$ and so place the corresponding Γ-markers back on line, we want to ensure that we are able to keep the later on their new position (and not let them roll further down) by C-restraining.*
   - Enumerate Γ-axioms for the arguments in $[k, \max_t(\gamma(t) \downarrow)]$. *This action puts the Γ-markers back on line and also activates the C-restraint of their new position.*

The module above functions as follows: when it is called for first time (or after an initialization) it starts from step 1. Each time it is called, we say that it executes one round. It starts a new round from the point it last stopped. If it has stopped on the end of some step, then it starts from the beginning of the next one (the next of step 3 is 1). In one round it can only execute one step. If it last stopped on a 'wait' instruction, in the next round it checks whether the relevant test is satisfied and waits further or moves on accordingly.

Note that according to the definition of Γ_k given above, *any marker rolling to a new position must have come (and been 'allowed' to roll down) from the next higher position.* The strategy for $\mathcal{P}$ is simply to hold a witness x from its use-set, larger than the restraint imposed on it and wait until $\Phi(x) \downarrow = 0$ (when it requires attention). Then it puts $x \searrow C$ and initialize higher priority ($\mathcal{Q}$-) requirements (and stops requiring attention). Assume an effective listing of all the requirements like

$$\mathcal{P}_0 > \mathcal{Q}_0 > \mathcal{P}_1 > \mathcal{Q}_1 \dots$$

Above we defined r_n (the n-restraint of a $\mathcal{Q}$-requirement) and the restraint to which a positive requirement listens, in the simpler case of a single $\mathcal{Q}$-requirement above (i.e. higher than) an infinite list of positive requirements. In the full case each $\mathcal{Q}$ requirement has its own n-restraints (defined in exactly the same way) and the restraints imposed on some $\mathcal{P}$ on the list are defined analogously. Namely $\mathcal{P}_t$ listens to the i-restraint for all $0 < i \leq (t-k)$, of $\mathcal{Q}_k$ for each $k < t$. Remember that there are no 0-restraints.

The restraint imposed on some $\mathcal{P}$ may change only finitely many times; and each time it changes we make sure that $\mathcal{P}$ is initialized (so that it picks up a new appropriate witness). In particular, whenever r_n of some $\mathcal{Q}_k$ changes value (according to the way we defined it) we assume that all positive requirements which listen to it, are initialized. In this case these are the $\mathcal{P}_i$ for $i \geq n+k$.

To sum up, positive requirements initialize the lower $\mathcal{Q}$ requirements, when they act. And once they've acted they don't act again and so each of them can only cause initialization at most once. When $\mathcal{Q}$ is initialized, it starts working anew (with a new, completely empty undefined Γ, γ etc.). A $\mathcal{Q}_k$ requirement causes initialization every time one of its n-restraints changes value; so it could cause initialization infinitely often. However, each of its r_n changes only finitely many times (since the residents of its n-th position stabilize). And since a change of r_n initializes only the $\mathcal{P}_i$ with $i \geq n+k$, each positive requirement is initialized finitely often. Notice that the steps in $\mathcal{Q}$'s module that can cause a change on its restraints are steps 1 and 3.

Construction At stage s we first successively access each of $\mathcal{Q}_i$ strategies for $i < s$ and run them (as described above). Then we choose the highest $\mathcal{P}$ which requires attention and satisfy it.

Notice that each $\mathcal{Q}$ involves infinitary activity and so it must be visited infinitely many times. A $\mathcal{Q}$ requirement only enumerates in its own set A and not any set (like C) related to other requirements. Also it can be initialized only finitely many times since it has finitely many $\mathcal{P}$ predecessors.

Verification First we need to show the following

Lemma 25. *Assume for some $\mathcal{Q}_k$ that $\Phi_k^C = W_k$. Then there are infinitely many positions which permanently stay outside the black area of $\mathcal{Q}_k$, and each of them has only finitely many (and at least one) permanent residents (i.e. Γ-markers). Also, for any position there is a stage beyond which it is not given additional Γ-markers.*

Proof. By induction: assume that it holds for the first $n-1$ positions outside the black area. Notice that z-positions on the real line are enumerated from right to the left. So, when the positions are still outside the black area, the ones with the smaller indices are on the right with respect to the ones with the bigger indices. Say that after s_0 no $\mathcal{P}_i$ with $i \leq n+k$ acts and no additional Γ-marker ever occupies one of the first $n-1$ positions (permanently) outside the black area.

Since $\Phi_k^C = W_k$, the module of $\mathcal{Q}_k$ doesn't get stuck on a 'wait' instruction and so it keeps on running its steps forever. After s_0 we keep on enumerating positions with initial residents successive arguments for which Γ (of $\mathcal{Q}_k$) was previously undefined. The markers sitting on the *current* n-th position after s_0 are not going to be moved. Indeed, according to the construction these markers are restrained from moving by r_n. And since no $\mathcal{P}_i$ with $i \leq n+k$ acts, r_n is not going to be violated anymore. For the same reason, r_{n+1} cannot be violated, and so no additional markers will move to the n-th position (coming from the $(n+1)$-th position). This completes the induction step. The base of the induction (i.e. the case for the 1-st position) is done in the same way, since after $\mathcal{Q}_k$ is initialized for the last time, r_1 is never violated. In particular, no Γ-marker can end up undefined. $\square$

Now suppose that $\Phi_k^C = W_k$. It follows from the construction and the above proof that Γ of $\mathcal{Q}_k$ is total. Indeed, axioms are being enumerated infinitely often, and the use γ for each of them remains the same throughout the construction. In particular, Γ is a wtt-reduction. It is also correct. Step 3 of $\mathcal{Q}$'s module ensures that any wrong computations are being corrected; and this is always possible since Γ-markers are always defined and they always sit on a number outside A.

As a result of lemma 25 and the definition of r_n, any n-restraint of a $\mathcal{Q}$ requirement reaches a limit. This means that each $\mathcal{P}$ requirement has only a finite restraint to deal with, and so it is eventually satisfied. This concludes the proof of the theorem. We would like to note that all representations A built in the above proof are (automatically, as a result of the construction itself) C-computable. So we actually build A within the C-ideal, as pictured in figure 4.2.

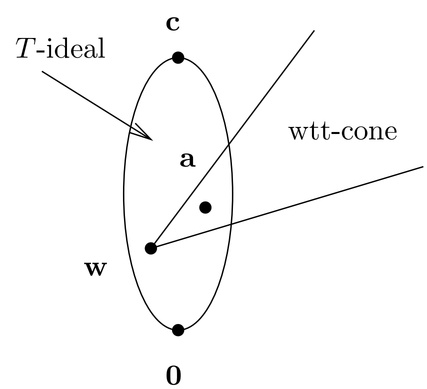

Figure 4.2: Degrees of Representations

Chapter 5

Randomness Digression

Algorithmic Randomness appeared in the mid-sixties with the pioneering work of Kolmogorov, Martin-Löf, Solomonof, Chaitin (there were some earlier more informal or less successful attempts by von Mises and Church, see [21] for historical background and references). It did not originate from the core of computability (recursion) theory, but it appeared in various other related fields as algorithmic probability, (program-size) complexity, effective measure theory and others. However, recently it has attracted considerable attention from computability theorists, who have advanced the subject by applying and developing techniques from the much more mature subject of computability. Randomness can be expressed in computability theoretic terms and it is interesting to see how this notion relates to traditional notions from computability theory; this note is in this spirit.

In the following we will use Solovay's characterisation of randomness (simplified by an observation of Downey[12]) presented as follows.

Definition 28. *A Solovay test is a c.e. set S of finite binary strings with $\sum_{\sigma\in S} 2^{-|\sigma|} < \infty$.*[1] *A Solovay test S captures a set A when for infinitely many $\sigma \in S$ it is the case that $\sigma \sqsubset A$.*[2] *The set A is random if there is no Solovay test which captures it.*

Here we identify a set A with its characteristic sequence and $\sigma \sqsubset A$ means that σ is a prefix of A. We think a Solovay test as a sequence of guesses about initial segments of a set A. What we require beyond the effectiveness in the enumeration is that our guesses are bold enough: this is the condition $\sum_{\sigma\in S} 2^{-|\sigma|} < \infty$ which assures that the length of the strings increases fast, and so the guesses are quite risky. In these terms, a set is random if we cannot guess infinitely often its initial segments by making bold enough

[1] $|\sigma|$ is the length of the string σ.

[2] in standard terminology it is said that A does not pass the test S.

guesses; in other words, if it is 'immune' against a particular kind of systems of guesses (called Solovay tests). Random sets are often said to be chaotic and unpredictable. The last refers to the difficulty we have to predict the next digits of its characteristic sequence, having an initial segment of it.

If we try to guess single elements of A instead of initial segments, then we arrive to the traditional in computability notion of immunity. Here a sequence of guesses is a computable sequence, or a strong or weak disjoint array. And immune (hyperimmune or hyperhyperimmune) sets are said to be quite sparse and complicated. One might even say that they are unpredictable —but this should be understood in a different way than in the case of randomness since here we don't assume we possess an initial segment of the set in order to predict following digits. However random sets should be expected to possess some immunity properties; for example it is not difficult to show that they are immune. And since A is random iff $\mathbb{N} - A$ is random (easy to see) they are bi-immune. In the following we show that they have even stronger immunity properties, but they cannot be hyperimmune. This strong negative property makes a set non-random, i.e. better behaved from the point of view of randomness.

Set $\Sigma = \{0, 1\}$ and Σ^* for the set of all finite binary strings; to any $\sigma \in \Sigma^*$ we assign the set $\sigma\Sigma^\omega = \{p \in \Sigma^\omega \mid \sigma \sqsubseteq p\}$ where Σ^ω is the set of infinite binary sequences. This way we get sets generating the usual topology in Σ^ω.

String means binary string, $\sqcup$ is disjoint union and since we identify sets with their characteristic sequence, by subset of an n-bit sequence we mean an n-bit sequence which has 0's in the positions where the initial sequence has 0's.

5.1 Randomness and immunity

Definition 29 (Fenner, Schaefer[17]). *A set A is called k-immune ($k \in \mathbb{N}$) when there is no strong disjoint array $(D_{g(n)})$ with $|D_{g(n)}| \leq k$ and $D_{g(n)} \cap A \neq \emptyset$ for all n. It is called ω-immune when it is k-immune for all k.*

Note that every set is 0-immune and that 1-immune are exactly the immune sets; also, h-immune sets are *omega*-immune. It is not difficult to show that the k-immune classes form a proper hierarchy (see Fenner,Schaefer[17]). The following result shows that this hierarchy is transcended not only by h-immune but also by random sets.

Theorem 19. *Random sets are ω-immune.*

For the proof we need the following

Lemma 26. *Given a set B with m elements and n disjoint subsets of it, say $(B_i)_{i=1,\dots n}$, of cardinality k, there are exactly $2^{m-kn} \cdot (2^k-1)^n$ subsets $D \subseteq B$ with $D \cap A_i \neq \emptyset$ for all i.*

Proof. Viewing finite sets as finite strings representing their characteristic sequence, we apply the counting principle. There are 2^k-1 non-empty subsets of B_i; so, there are $(2^k-1)^n$ subsets $C \subseteq \cup_i B_i$ with $C \cap B_i \neq \emptyset$ for all i. Finally, since $|B - \cup_i B_i| = m - kn$, there are $2^{m-kn} \cdot (2^k-1)^n$ subsets $D \subseteq A$ with $D \cap B_i \neq \emptyset$ for all i. □

Proof of the theorem. For a contradiction, suppose that A is random and not k-immune. There is disjoint strong array $(D_{f(s)})$ with $D_{f(s)} \cap A \neq \emptyset$ and $|D_{f(s)}| = k$ for all s. W.l.o.g. assume that the elements in D_s are all smaller than these in D_{s+1}. Using $(D_{f(s)})$ we will define a Solovay test which captures A. Define $g(s) = \max D_{f(s)}$.

At stage s we output all the $g(s)$-bit strings which intersect each of $(D_{f(i)})_{i=1,\dots s}$. According to lemma 26 there are $2^{g(n)-kn} \cdot (2^k-1)^n$ such strings (take the sequence $1^{g(n)}$ for B and $D_{f(i)}$ for B_i). So, if I_s is the set of strings we enumerate at stage s, it is the case that

$$|I_s| = 2^{g(s)-ks} \cdot (2^k-1)^s$$

and $I = \sqcup_{s>0} I_s$ is a Solovay test. Indeed,

$$\sum_{\sigma \in I} \mu(\sigma\Sigma^\omega) = \sum_{\sigma \in I} 2^{-|\sigma|} = \sum_{s>0} \sum_{\sigma \in I_s} 2^{-|\sigma|} = \sum_{s>0} 2^{g(s)-ks} \cdot (2^k-1)^s \cdot 2^{-g(s)}$$
$$= \sum_{s>0} 2^{-ks} \cdot (2^k-1)^s = \sum_{s>0} (1-2^{-k})^s = 2^k - 1 < \infty.$$

Finally it is clear from the construction that for any s there is a string $\sigma \in I_s$ with $\sigma \sqsubset A$; since the sets I_s are disjoint, there are infinitely many $\sigma \in I$ with $\sigma \sqsubset A$. So A is not random. □

From theorem 19 we get the well known fact that if x is a random sequence then for any $n \in \mathbb{N}$ there are infinitely many n-bit blocks of 0's in x. We can also define a wider hierarchy than that of k-immune sets if instead of strong array in definition 29 we say weak array. If we call the corresponding sets strongly k-immune, a slight modification of the above proof shows that random sets are strongly k-immune for all k.

Theorem 20. *No random set is hyperimmune.*

Proof. For a contradiction, suppose that A is random and hyperimmune. Consider the strong array $D_{f(0)} = \{1\}$, $D_{f(1)} = \{2,3\}$, $D_{f(2)} = \{4,5,6,7\}$, ... which is formally defined as

$$\begin{aligned} D_{f(0)} &= \{1\} \\ D_{f(n+1)} &= \{\max D_{f(n)} + 1, \dots, \max D_{f(n)} + 2 \cdot |D_{f(n)}|\}. \end{aligned}$$

So $D_{f(n)} = 2^n = 2 \cdot |D_{f(n-1)}|$, and $\min D_{f(n)} = \max D_{f(n-1)} + 1 = \sum_{i=0}^{n-1} 2^i + 1 = 2^n$. Since A is hyperimmune there are infinitely many n with $D_{f(n)} \cap A = \emptyset$. We will define a Solovay test S which captures A.

At stage s, enumerate in S all the strings of length 2^{s+1} whose last $|D_{f(s)}| = 2^s$ digits are 0. These are 2^{2^s} strings of length $2 \cdot 2^s = 2^{s+1}$. To show that S is a Solovay test, let $(\sigma_{si})_{i<2^{2^s}}$ be a 1-1 enumeration of the strings enumerated in S at stage s. We have

- $S = \sqcup_{s\in\mathbb{N}}\{\sigma_{si} \mid i < 2^{2^s}\}$
- $\mu(\sigma_{si}) = 2^{-|\sigma_{si}|} = 2^{-2 \cdot 2^s}$

and so

$$\sum_{\sigma\in S} 2^{-|\sigma|} = \sum_{s\in\mathbb{N}} \sum_{i<2^{2^s}} 2^{-|\sigma_{si}|} = \sum_{s\in\mathbb{N}} 2^{-2\cdot 2^s} \cdot 2^{2^s} = \sum_{s\in\mathbb{N}} 2^{-2^s} < \sum_{s\in\mathbb{N}} 2^{-s} < \infty$$

Consider an increasing sequence $(s_i)_{i\in\mathbb{N}}$ of stages such that $D_{f(s_i)} \cap A = \emptyset$. This means that at stage s_i one of the strings enumerated in S is a prefix of A. Indeed, considering A as a binary sequence, in the positions $r \in D_{f(s_i)}$ there are 0's; so that the last $|D_{f(s_i)}| = 2^{s_i}$ digits agree with $A \upharpoonright 2^{s_i+1}$. And for the first 2^{s_i} digits we have considered all cases. Since all strings enumerated are distinct and (s_i) is infinite, the test S captures A and so the last is not random. □

5.2 Difference hierarchy and randomness

Theorem 21. *If $\sum_n f(n)2^{-n} < \infty$ then there is no f-c.e. random set; but the converse does not hold.*

Proof. Let A be an f-c.e. set with $\sum_n f(n)2^{-n} < \infty$ and

$$a[s] = a_{0s}a_{1s}a_{2s}\ldots$$

($a_{ns} \in \{0,1\}$ and $\lim_s a_{ns}$ exists) an f-approximation to the characteristic sequence of A in the sense that $n \in A \iff \lim_s a_{ns} = 1$ and $|\{s \mid a_{ns} \neq a_{n,s+1}\}| \leq f(n)$; this means that we can change our mind about whether $n \in A$ at most $f(n)$ times. Say also that A is not computable since this case is trivial. W.l.o.g. we can assume that for any fixed s there is exactly one n with $a_{ns} \neq a_{n,s+1}$.

We define a Solovay test S as follows: at stage s, if $a_{ns} \neq a_{n,s+1}$ we enumerate the string $\sigma_s = a_{0s}\ldots a_{ns}$. It is easy to see that there are infinitely many s such that $a_{ns} \neq a_{n,s+1}$ for some n and $A \restriction n+1 = A[s+1] \restriction n+1$ (infinitely many non-deficiency stages)[3]. This means that there are infinitely many s with $\sigma_s \sqsubset A$. Also

$$\sum_{\sigma\in S} 2^{-|\sigma|} = \sum_m \sum_{\substack{\sigma\in S\\|\sigma|=m}} 2^{-|\sigma|}.$$

But since $a[s]$ is an f-approximation of A we have $|\{\sigma \in S \mid |\sigma| = m\}| < f(m)$ for all m. So

$$< \sum_m f(m)\cdot 2^{-m} < \infty$$

by hypothesis. So S is a Solovay test which captures A, and the last is not random.

That the converse does not hold follows from the fact that there are computable f with infinitely many zeros and $\sum_n f(n)2^{-n} = \infty$. Indeed, in that case we can effectively find the zeros of f, and if we have an f-approximation of a set A we can construct an infinite increasing sequence (n_k) and a program g such that $g(n_k) = 1 \iff n_k \in A$. This means that A is not bi-immune, and so not random. □

Corollary 4. *If f is bounded by a polynomial, then there are no random f-c.e. sets.*

Proof. By the assumption we have $\sum_n \frac{f(n)}{2^n} < \infty$ and so the result. □

Note that every random c.e. real is f-c.e. as a set, for $f(n) = 2^n$.

[3] Here $A[s]$ is the s-approximation of A, i.e. the set corresponding to the characteristic sequence $a[s]$.

Chapter 6

Hypersimplicity and Semicomputability in the Weak Truth Table Degrees

6.1 Introduction

We are interested in how hypersimplicity and semicomputability (in the sense of Jockusch [19]) relate to the weak truth table degrees. Hypersimple sets where invented by Post as a solution to his problem (now called Post's problem) for the structure of truth table degrees. Then they where shown to be a natural solution to Post's problem for the weak truth table degrees as well. So it is interesting to know the distribution of these natural solutions in the weak truth table degrees. Moreover, weak truth table reducibility is the most appropriate for the study of hypersimplicity given that its essense is the existense of computable bounds (in the use of the relative computation) and hypersimplicity of a set A is based on the same notion: a computable sequence of bounds f(n) below which we get (strictly) more and more elements outside A. This connection becomes even more clear if we note that elements outside A below computable bounds are also important in a weak truth table reducibility since only elements not yet in A and below the use can rectify computations.

It is known (Jockusch[19]) that every c.e. wtt degree has a c.e. semicomputable member while an old theorem of Post asserts that the complete wtt degree contains no hypersimple set. The latter proof makes full use of the completeness of the halting problem. In the next section we show that the c.e. wtt degrees which are bounded by no hypersimple degree (a property of the complete degree) are quite common. In particular, they occur outside any non-trivial cone of Turing degrees. The existence of such sets can be intuitively justified as hypersimple sets have quite 'sparse' compliments while a wtt reduction $A \leq_{\text{wtt}} W$ in general requires numbers of *fixed* segments of $\mathbb{N}$ to stay outside W (in order to be used for the rectification of the functional we are building, if needed).

Next, we show that the hypersimple-free c.e. wtt degrees are downwards dense in the c.e. wtt degrees; i.e. every non-zero c.e. wtt degree bounds a non-zero hypersimple-free c.e. wtt degree. We ask whether this can be extended to full density and we conjecture a negative answer. Furthermore, we show that for every hypersimple wtt degree there is one strictly above it.

In the final section we study the wtt degrees which contain sets that are both hypersimple and semicomputable. We characterize this class as the c.e. cuts of computable linear orderings of $\mathbb{N}$ of order type $\omega + \omega^*$ (where ω^* is the inverse of ω). This characterization will help a lot in the constructions involving such sets as we only have to deal with linear orderings with the *finite predecessor-or-successor* property (that is, each number has either finitely many predecessors or finitely many successors) and not with a conjunction of hypersimplicity and semicomputability.

Using this, we point out that the wtt *degrees of approximation representations for c.e. reals* studied in chapter 3, 4 are exactly the wtt degrees of hypersimple semicomputable sets (in fact the actual classes of sets coincide) and so some of the results there can be stated in terms of the present chapter and contribute to our study. For example, *there is a hypersimple wtt degree which is bounded by no hypersimple semicomputable degree* (a corollary of a result in chapter 4). Moreover, we can consider the c.e. wtt degrees decomposed into two classes: the ones that are bounded by a hypersimple semicomputable degree and the ones that are not. Since the first one is downwards closed and the second is upwards closed we can think of them as the bottom and upper part of the c.e. wtt degrees (with respect to this decomposition). The two classes are non-trivial (as it follows from chapter 4) and two very interesting questions are

(a) Are there minimal elements of the upper class?

(b) Are there maximal elements of the bottom class?

A positive answer to question (a) would mean the existence of a bottom of a hypersimple semicomputable free upper cone in the c.e. wtt degrees which bounds only elements of the first class. A positive answer to question (b) would mean the existence of maximal hypersimple semicomputable wtt degrees (in the sense that no degree above them is hypersimple semicomputable). In the last section we prove that there is no maximum hypersimple semicomputable wtt degree (theorem 26). Moreover we construct two degrees of the bottom class whose join belongs to the upper class. This shows that the bottom class is not an ideal and the hypersimple semicomputable wtt degrees are not closed under join. We wish to note that most of the proofs in this chapter do not rely on classical strategies for the satisfaction of the requirements. For example, in theorem 26 we are building a set avoiding a given initial segment in the c.e. wtt degrees but the usual Sacks coding cannot be applied because of the nature of the sets we are dealing with. So we needed to design a strategy based on the fact that we are dealing with hypersimple semicomputable sets.

In the following we use standard notation and when we describe a construction we assume a *current value* (corresponding to the current stage) for each of the various parameters involved. All the degrees will be c.e. and $A \leq_{\text{wtt}} B$ is indicated as $\Phi^B = A; \phi$ when we wish to make the algorithm (functional) Φ and the computable use ϕ of the reduction explicit. Finally, we use ℓ to denote the length of agreement of a potential reduction e.g. $\ell(\Phi^W = A; \phi)$ is the length of agreement of $A \leq_{\text{wtt}} W$ via the functional Φ and with use bounded by the partial computable function ϕ.

6.2 Wtt c.e. degrees that are not bounded by hypersimple wtt degrees

In this section we look at wtt c.e. degrees that are not bounded by hypersimple wtt degrees. These are degrees containing c.e. sets that cannot be wtt-coded into hypersimple sets. In other words they are bottoms of hypersimple-free upper cones in the wtt degrees.

Theorem 22. *Wtt c.e. degrees that are not bounded by hypersimple wtt degrees occur outside any non-trivial upper cone of c.e. Turing degrees. Formally, if B is c.e. and non-computable then there exists $A \not\geq_T B$ such that the upper cone $\{\mathbf{w} \mid \mathbf{w} \geq \mathbf{a}\}$ in the c.e. wtt degrees is hypersimple-free.*

Proof. Apart from $A \not\geq_T B$ which can be achieved in a standard way (via Sacks restraints) the requirements we have to satisfy are

$$\mathcal{Q}_{\Phi,W} : \Phi^W = A; \phi \Rightarrow \begin{cases} \exists (D_n)((D_n) \text{ sequence of consecutive segments of} \\ \mathbb{N} \wedge \ \forall n(\overline{W} \cap D_n \neq \emptyset)) \end{cases}$$

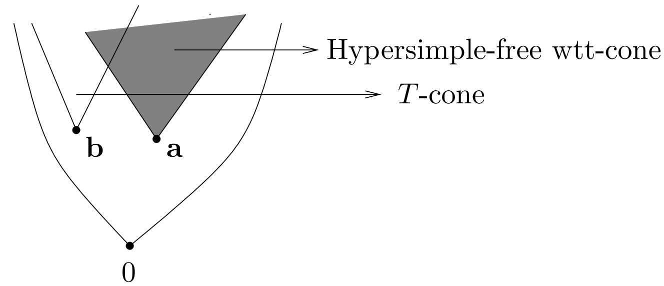

Figure 6.1: Theorem 22.

As usual, we can assume that ϕ is strictly monotone. The effective sequence (D_n) above will serve as a disjoint array witnessing that W is not hypersimple. If $\forall n(\overline{W} \cap D_n \neq \emptyset)$ fails, we will be able to diagonalize successfully against $\Phi^W = A; \phi$; this will be achieved via a ripple of diagonalizations, the last of which is successful (i.e. is not rectified).

The strategy for $\mathcal{Q}_{\Phi,W}$ consists of steps A_n, B_n, $n \in \mathbb{N}$. The family $(A_n)_{n\in\mathbb{N}}$ enumerates (D_n). If at some stage we find that (D_n) does not fulfil the purpose of its construction (i.e. $D_n \subseteq W$ occurs for some n) we turn to step B_n (for that particular n that witnessed the failure). This D_n-failure step will start a ripple of diagonalizations, succeeding $\Phi^W \neq A; \phi$. Hence, either all A_n are performed (thus satisfying $\mathcal{Q}$ via its second clause) and no B_n is activated, or finitely many A_n are performed, until a single B_n-step is activated which (eventually) ends $\mathcal{Q}$'s activity (satisfying it through the negation of its first clause). $\mathcal{Q}$ will choose the witnesses for its diagonalizations from a special set $U \subseteq \mathbb{N}$ disjoint from the special sets of other requirements (e.g. $U = \mathbb{N}^{[e]}$, the e-th column of $\mathbb{N}$ where e is the index of $\mathcal{Q}$ under an effective ordering of the requirements). Let $a_1 = 1$ and $I_0 = \emptyset$. The A_n, B_n steps are as follows:

A_n *(D_n definition)*

1. Define I_n as the set of the next a_n unused (i.e. not in $\cup_{i<n} I_i$) elements in U. *This is the set of witnesses (agitators) of step A_n. They have the potential to be used by B_n after an A_n-failure. Their number $|I_n| = a_n$ is defined by the previous step A_{n-1}.*
2. Restrain I_n (from A) and wait until $\ell(\Phi^W = A; \phi) > t$, for all $t \in I_n$.
3. Define $D_n := \{\max D_{n-1}+1, \dots, \max_{i\in I_n} \phi(i)-1\}$ and $a_{n+1} := |\cup_{i\leq n} D_i|+1$ $(= \max \phi(\cup_{i\leq n} I_i) + 1)$.

B_n *(D_n-failure diagonalization loop)*

(a) Wait for a Φ-expansionary stage.

(b) Put the least element of $I_n \cap \overline{W}$ into W and go to (a).

The $\mathcal{Q}$-module operates as follows: it executes $A_1, A_2, \dots$ but before moving to A_n it checks whether $D_i \cap \overline{W} \neq \emptyset$ for all $i < n$. If this holds, it proceeds to A_n, otherwise it proceeds to B_k for the least k with $D_k \cap \overline{W} = \emptyset$. When the $\mathcal{Q}$ module is called, it starts operating from where it last stopped, until it meets a 'wait' condition which is not fulfilled *or* it finishes an A_n step (in which case it stops at the beginning of A_{n+1}). We start from A_1.

Now that we have defined the operation of $\mathcal{Q}$, we explain why this strategy works. First of all note that $D_1, D_2, \dots$ are consecutive segments of $\mathbb{N}$ and $I_1, I_2, \dots$ are consecutive segments of U (the use-set of $\mathcal{Q}$). The restraints set on U are potentially infinite, but this is no problem as numbers in U are only used by $\mathcal{Q}$. The outcomes are as follows:

1. when $\mathcal{Q}$ executes all A_n. Then, according to the module (D_i) is an infinite disjoint array with $D_i \cap \overline{W} \neq \emptyset$ for all i. Indeed, in order to proceed to A_{n+1} we must make sure that $D_i \cap \overline{W} \neq \emptyset$ for all $i \leq n$.

2. when we are permanently stuck in a 'wait' instruction in some A_n step. In this case it is obvious that $\Phi^W \neq A; \phi$ and $\mathcal{Q}$ is satisfied.

3. when the above fail, and so the $\mathcal{Q}$-module passes control to some B_n step. This must happen after $D_n \cap \overline{W} = \emptyset$ (i.e. $D_n \subseteq W$) has been noticed by the module.

In the third outcome, B_n will start a ripple of at most $|I_n|$ diagonalizations and we claim that the last one will be impossible to rectify. In other words that $\Phi^W \neq A; \phi$ is a certain final outcome. Indeed, the only rectification codes (i.e. numbers that can rectify Φ^W computations) for any agitator in I_n are in $\mathbb{N} \restriction \max \phi(I_n)$ and so they are not more than $\max \phi(I_n)$. But $\mathbb{N} \restriction \max \phi(I_n) = \cup_{i \leq n} D_i$ and since $D_n \subseteq W$ any rectification code (for witnesses in I_n) is in $\cup_{i<n} D_i = \mathbb{N} \restriction \max \phi(I_{n-1})$. So if R_n is the set of these codes,

$$|R_n| = \max \phi(I_{n-1}) < \max \phi(I_{n-1}) + 1 = a_n = |I_n|.$$

Since for each I_n-enumeration (into A, at a Φ-expansionary stage) at least one R_n-enumeration (into W) is needed for a new expansionary stage to come, there will be a (final) I_n diagonalization which is not rectified. This means that the module will be stuck on (a) of B_n unable to obtain an expansionary stage. So $\Phi^W \neq A; \phi$ and the third outcome satisfies $\mathcal{Q}$. Hence the module is successful.

The *construction* for the satisfaction of the $\mathcal{Q}$ requirements is: at stage s run successively the modules of $\mathcal{Q}_0, \dots, \mathcal{Q}_s$. The satisfaction of the requirements follows by the analysis of outcomes we discussed above. In particular, there is no injury. If we wish to add the requirement $A \not\geq_T B$ for some given c.e. non-computable B we just need to attach the $\mathcal{Q}$ requirements in the usual Sacks-restraint argument (e.g. on a tree) for the satisfaction of:

$$\mathcal{N}_\Phi : B \neq \Phi^A.$$

There is no non-trivial interaction of strategies apart from those discussed above and those in the classical Sacks argument. Each $\mathcal{Q}$ strategy will occupy a (1-branching) node on the tree and will only be asked to respect a finite amount of A-restraint. So the only modification in its strategy is to choose I-witnesses larger than this finite (or at least with $\liminf < \infty$) restraint. The verification of this construction follows the lines of the classical Sacks argument and our analysis of outcomes for the $\mathcal{Q}$-requirements. $\square$

In chapter 4 we constructed a hypersimple set which is not wtt-bounded by any cut any computable ordering of $\mathbb{N}$ of order type $\omega + \omega^*$. By theorem 25 of section 6.5 this implies (in fact, is equivalent to)

Corollary 5. . *There is a hypersimple set which is $\leq_{wtt}$-bounded by no set which is both hypersimple and semicomputable.*

We would like to make an interesting comparison between the $\mathcal{Q}$-strategy in the proof of theorem 22 with the strategy employed in chapter 4 in order to prove the previously mentioned version of corollary 5. The crucial difference is that in the latter, the A-restraint on columns of $\mathbb{N}$ (imposed by a fixed requirement) is only finite; and this is what allows us to make A hypersimple. Here is how we achieve this: our typical requirement is

$$\mathcal{Q}'_{\Phi,W,\psi} : \Phi^W = A; \phi \Rightarrow \begin{cases} W \text{ is not the left cut of the computable} \\ \text{ordering of } \mathbb{N} \text{ of order type } \omega+\omega^* \text{ defined by } \psi \end{cases}$$

Here ψ is the function possibly defining such an ordering $\prec$ on $\mathbb{N}$ (in the sense that $\psi(n,m) = 1 \iff n \prec m$) with left cut W; Φ runs over the partial computable functionals, ϕ, ψ over the partial computable functions and W over the c.e. sets. In a family of steps (A_n) (similar to the ones we used above) we enumerate a set D (instead of an array as in the above argument) intended to be $\overline{W}$. If at some point $D \cap W \neq \emptyset$

we are able to diagonalize through a B_t step in a way analogous to the above proof. This way we are able to satisfy the following requirements:

$$\mathcal{Q}''_{\Phi,W,\psi} : \Phi^W = A;\phi \Rightarrow \begin{cases} W \text{ is not the left cut of the computable} \\ \text{ordering of } \mathbb{N} \text{ of order type } \omega+\omega^* \text{ defined by } \psi \\ \text{or } \overline{W} = D \text{ (so } W \text{ is computable).} \end{cases}$$

The satisfaction of all $\mathcal{Q}''$ imply that A is non-computable. Using this, $\mathcal{Q}''$ implies $\mathcal{Q}'$. Moreover, the only way to have infinite restraints on $\mathcal{Q}'$'s column is to let the sequence (A_n) act forever. According to that construction, this implies that $D = \overline{W}$ and so W is computable. It also implies that $\Phi^W = A;\phi$ and hence the outcome '$D = \overline{W}$' is never realized (so we call it pseudo-outcome). Hence no sequence (A_n) acts forever and the restraint on columns of $\mathbb{N}$ imposed by any fixed requirement is only finite. Using this fact we are able to show that the hypersimplicity requirements are satisfied as well.

So the point is that in chapter 4, due to the special nature of the requirements we were able to force a stop on the (A_n) routine (and so, the restraint it imposes to the lower hypersimplicity requirements) whereas in the proof of theorem 22 we are not.

6.3 Hypersimple-free c.e. wtt-degrees

The next result shows that the c.e. wtt hypersimple-free degrees are more common than the ones studied in the previous section. In fact, we show their downward density in the c.e. wtt degrees.

Theorem 23. *The hypersimple-free c.e. wtt-degrees are downwards dense in the c.e. wtt-degrees. That is, if* $\mathbf{c} > \mathbf{0}$ *then there is a c.e. hypersimple-free* $\mathbf{a}$ *such that* $\mathbf{0} < \mathbf{a} < \mathbf{c}$.

Proof. By the density of the c.e. wtt-degrees it is enough to show that for every $\mathbf{c} > \mathbf{0}$ there is a hypersimple free $\mathbf{a}$ with $\mathbf{0} < \mathbf{a} \leq \mathbf{c}$. Suppose a non-computable c.e. set C. We are going to construct a non-computable c.e. $A \leq_{\text{wtt}} C$ and equivalent to no hypersimple set. The requirements (apart from the permitting $A \leq_{\text{wtt}} C$) are:

$$\mathcal{Q}_{\Phi,\Psi,W} : \Phi^W = A;\phi \text{ and } \Psi^A = W;\psi \Rightarrow \begin{cases} \exists(D_n)((D_n) \text{ sequence of} \\ \text{consecutive segments of } \mathbb{N} \wedge \\ \forall n(\overline{W} \cap D_n \neq \emptyset)) \end{cases}$$

We also have the non-computability requirements

$$\mathcal{P}_\Phi : A \neq \Phi.$$

We start off with the following atomic module for $\mathcal{Q}_{\Phi,\Psi,W}$. The idea behind this strategy is similar to the one of theorem 22: assuming $\Phi^W = A;\phi$ we enumerate a strong array (D_n) and try to achieve $\overline{W} \cap D_n \neq \emptyset$ for all n. The definition of each D_n is such that if we all of its elements appear in W later on (giving $\overline{W} \cap D_n = \emptyset$) then we are able to ensure $\Phi^W \neq A;\phi$ by diagonalizing. But since we want $A \leq_{\mathrm{wtt}} C$ such diagonalization must be C-permitted. So since C is arbitrary, in general it will not allow the number of diagonalizations that steps B_n performed in theorem 22. To avoid this difficulty we modify the enumeration of (D_n) using the additional hypothesis $\Psi^A = W;\psi$ that we are given and we make sure that if $D_n \subseteq W$ occurs then we are able to destroy $\Phi^W = A;\phi$ with *a single diagonalization*. Hence, every D_n definition is associated with a diagonalization witness a which will be used if and when $D_n \subseteq W$ occurs.

The C-permitting is represented formally by a function (in other words, a functional with empty oracle) Δ which tries to compute C. Let U an infinite computable set especially for the use of $\mathcal{Q}$ strategy. Below, s is the current stage and any parameters mentioned in the construction are supposed to have a current value.

A_n *(n-attack setup)* Find a least $a < s$ such that $a \in U - A$ and

- $\ell(\Phi^W = A;\phi) > a$
- for all $x \in \cup_{i<n} D_i(\psi(x) < a)$

and define $a_n = a$ and $D_n = \{\max \cup_{i<n} D_i + 1, \dots, \phi(a_n)\}$.

When A_n is run D_i for $i < n$ are already defined. If $D_n \subseteq W$ later on, we will be able to diagonalize successfully by $a_n \searrow A$ and imposing a finite restraint on A (in order to preserve a segment of W).

B_n *(D-failure step; in particular when the D-enumeration done in A_n has proved wrong)*

Consider a_n which was defined in step A_n.

(a) Wait until $\ell(W, \psi^A; \psi) > x$ for all $x \in \cup_{i<n} D_i$. Restrain $A \upharpoonright v$ where v is the use of these computations.

(b) Express desire for $a_n \searrow A$: define the functional $\Delta \upharpoonright a_n = C \upharpoonright a_n$.

> *It will be $v < a_n$ as in step A_n. If we are permitted to put $a_n \searrow A$, this enumeration respects the A restraint we imposed in (a) above. This diagonalization can only be rectified via a W-enumeration below $\Phi(a_n)$. But no such enumeration can happen with elements in $\cup_{i<n} D_i$ due to the A restraint we impose. Hence it must be with elements in D_n; however these are already in W (this made us start step B_n) and so the disagreement we create is permanent.*

The parts A_n, B_n above are only a piece of the whole $\mathcal{Q}$-strategy. We call them AB-routine. A recursive iteration of AB -routines $(AB(0), AB(1), \dots)$ constitutes the $\mathcal{Q}$-strategy. We explain how a single AB-routine works. It enumerates its own array (D_n), which is a sequence of consecutive segments of (and potentially covering) $\mathbb{N}$, while Δ belongs to all AB-routines. It starts by performing successively the steps $A_1, A_2, \dots$ and at each A_i it defines D_i. It also finds a suitable a_i which is a witness for a 'back-up diagonalization' planned in case $D_i \subseteq W$ later on, i.e. in case the guess made in A_i is wrong.

After that A_i has been completed and in order to pass to A_{i+1} we check whether $D_k \subseteq W$ holds for some $k \leq i$. In other words, whether all the $\overline{W}$-guesses we made so far (via (D_k)) look correct. If yes, then we can proceed to A_{i+1} in order to push (D_k) further. Otherwise for the least n with $D_n \subseteq W$ we pass control to B_n. No more steps apart from B_n will ever be performed in this AB-routine. B_n activates the back-up diagonalization prepared in A_n: it suggests (at some later suitable stage) a_n as a witness for $\Phi^W \neq A; \phi$ and it also restrains the W-use of the computation (even after a possible $a_n \searrow A$). Note that in the atomic module above there are 'wait' instructions. Taking into account that we may have to wait forever, the outcomes of an AB-routine are:

$\boxed{1}^{AB}$ As we go through $A_1, A_2, \dots$ we get stuck in a 'wait' instruction of some A_i and stay there forever. According to the 'wait' conditions, this implies the satisfaction of $\mathcal{Q}$.

$\boxed{2}^{AB}$ Before passing to a next A_i we collapse onto a B_n-step. This does not automatically imply satisfaction of $\mathcal{Q}$ but it advances the functional Δ (which belongs to all AB-routines). If the Δ-axioms enumerated by B_n are later shown to be wrong, C will permit a_n and $\mathcal{Q}$ will be permanently satisfied.

$\boxed{3}^{AB}$ We go through $A_1, A_2, \dots$ with no permanent distraction. Under this outcome the AB-routine produces an infinite disjoint array (D_n) with $D_n \cap \overline{W} \neq \emptyset$ for all n, thus proving that W is not hypersimple (and $\mathcal{Q}$ is satisfied).

Turning to the whole $\mathcal{Q}$-strategy, we start executing $AB(1)$ (which is identical to the typical AB-routine described above) and continue as follows (in an inductive mode). If and when $AB(i)$ has come to an end (in the sense of outcome $\boxed{2}^{AB}$) *and* Δ looks correct, we start $AB(i{+}1)$ with the additional (but not essential) restriction that all the a_t-witnesses chosen during its A_t-steps are larger than the witnesses already suggested by the $AB(m)$ for $m < i$, when they terminated (this ensures that every time we pass to a higher AB-routine, Δ has grown longer). If Δ does not look correct, we finish with the $\mathcal{Q}$-*termination routine*:

(a) Let $n = \mu i[\Delta(i) \neq C(i)]$ and a the least witness $> n$ suggested at a previous AB-termination.

(b) Put $a \searrow A$ thus satisfying $\mathcal{Q}$

(the disagreement will be preserved as explained above). From the above, any enumeration into A is C-permitted and so $A \leq_{\mathrm{wtt}} C$. Note that as we go through $AB(1), AB(2), \ldots,$ we build on more and more restraints on A. If C is indeed non-computable, Δ must fail and so at some point the $\mathcal{Q}$-termination routine will satisfy $\mathcal{Q}$. The outcomes of the entire $\mathcal{Q}$-strategy are:

$\boxed{1}^{\mathcal{Q}}$ As we go through $AB(1), AB(2), \ldots$ we get stuck in a 'wait' instruction of some $AB(i)$ and stay there forever. Or some $AB(i)$ never stops running. Either case implies the satisfaction of $\mathcal{Q}$ as before, and also that the overall A-restraints that $\mathcal{Q}$ imposes are bounded (i.e. finite).

$\boxed{2}^{\mathcal{Q}}$ A Δ-check finds Δ wrong and we enter the $\mathcal{Q}$-termination routine. Again $\mathcal{Q}$ is satisfied as explained above.

$\boxed{3}^{\mathcal{Q}}$ We never stop running $AB(1), AB(2), \ldots.$ This means that Δ is total and correct, so that C is computable.

These outcomes show that our strategy is successful. Moreover it is not difficult to see that all $\mathcal{Q}$ strategies can work together with only a finite injury effect. Whenever some $\mathcal{Q}$ act it initializes all lower requirements and probably increases its A-restraints. But according to the outcomes above it acts only finitely often (imposing a final finite A-restraint) and so it allows lower priority requirements (which respect the higher priority A-restraint) to be satisfied. This also shows that the $\mathcal{P}$ requirements can be added with the same finite injury effect. We reserve special sets P for the witnesses of each $\mathcal{P}$ and let them act according to the usual non-computability strategy: choose a witness larger than the restraints of higher priority $\mathcal{Q}$ requirements, wait until $\Phi(t)\downarrow= 0$ and put $t \searrow A$. When $\mathcal{P}$ acts it initializes all lower priority requirements. When itself is initialized, it starts anew (with a new witness).

Construction At stage s let the highest $\mathcal{Q}$ or $\mathcal{P}$ requirement (with index $< s$) requiring attention act. A $\mathcal{Q}$ requires attention is one of its AB-routines requires attention; and this happens for $AB(i)$ if all higher AB routines have finished a B-step and itself is ready to move on a further step (after we successfully complete a Δ-correctness check, in case $AB(i)$ is at the beginning). Once an AB-routine ends up in a B-step it starts carrying the responsibility for the correctness of a segment of Δ (namely from the threshold marking the arguments on which the higher AB-routines have enumerated axioms, up to the largest argument for which $AB(i)$ enumerated computations). If a Δ-correctness check fails, we go back to the AB-routine which has the relevant responsibility, and in particular its B-step which enumerated the axioms.

This concludes the description of the construction. For the verification we note that (as explained above) any $\mathcal{Q}$ acts at most finitely many times and so all requirements can work together with the standard *finite injury* effect. The satisfaction of a single $\mathcal{Q}$ is already explained above and this is enough for the verification as there are no non-trivial interactions between the $\mathcal{Q}$ requirements. □

It is natural to ask whether downward density can be extended to full density of the hypersimple-free wtt degrees in the c.e. wtt degrees. If we start with an interval $B <_{\mathrm{wtt}} C$ (instead of just $\emptyset <_{\mathrm{wtt}} C$) one can see that the B-coding into A that we are constructing forces the need for multiple enumeration (similar to the diagonalization ripple of theorem 22) for the satisfaction of (the analogue of) $\mathcal{Q}$; and this requires multiple permitting by C which is not always available. So we conjecture that a non-density result may be possible.

6.4 Hypersimple Sets in the wtt Degrees: no maximal elements

The following theorem shows that there are no maximal hypersimple wtt degrees i.e. for every hypersimple wtt degree there is one strictly above it.

Theorem 24. *If W is hypersimple, there exists a hypersimple set A such that $W <_{wtt} A$.*

Proof. We have seen in the previous sections that there is a certain type of conflict when we try to construct a hypersimple set A above a given W, and sometimes this makes such a construction impossible. We show now that when we have the information that W is hypersimple, this conflict is managable and a construction is possible. If D_n is an effective enumeration of all finite sets and (Φ, ϕ) runs over an effective enumeration of all partial computable functionals/functions then the following requirements guarantee the result:

$$
\begin{aligned}
\mathcal{Q}: &\quad W \leq_{\mathrm{m}} A \\
\mathcal{P}_{\Phi\phi}: &\quad \Phi^W \neq A; \phi \\
\mathcal{R}_\phi: &\quad \exists n (D_{\phi(n)} \subseteq A) \ \vee \ D_\phi \text{ not a strong array.}
\end{aligned}
$$

We say that D_ϕ is a strong array if ϕ is computable and for $n \neq m$, $D_{\phi(n)} \cap D_{\phi(m)} = \emptyset$. Notice that $\mathcal{Q}$ asks for something stronger than we really need, namely m-reducibility instead of wtt. Fix a computable $c : \mathbb{N} \to \mathbb{N}$ which is 1-1 and such that $\mathbb{N} - c(\mathbb{N})$ is infinite (e.g. $c(n) = 2n + 1$). We will arrange that

$$n \in W \iff c(n) \in A$$

thus satisfying $\mathcal{Q}$. Assume a priority list where $\mathcal{Q}$ has highest priority and the infinitely many $\mathcal{P}_\Phi$, $\mathcal{R}_\phi$ follow in an effective way (based on the effective enumeration of (Φ, ϕ) that we assumed earlier). Each of $\mathcal{P}_\Phi$, $\mathcal{R}_\phi$ will be finitary (i.e. act finitely often) and any A-enumeration the do must not bring $\mathcal{Q}$ in difficult position. An A-enumeration affects $\mathcal{Q}$ only when it involves c-codes, i.e. elements in $c(\mathbb{N})$.

$\mathcal{P}_{\Phi\phi}$ **strategy** As usual, we can assume that ϕ is strictly monotone. We are going to attack $\mathcal{P}$ by steping on the hypersimplicity of W: we construct a strong array (F_n) which tries to show that W is not hypersimple, in such a way that when it fails (i.e. $F_n \subseteq W$) we are able to diagonalize successfully (i.e. in a way that makes a final dissagreement unavoidable) against $\Phi^W = A; \phi$. This will be achieved via a ripple of diagonalizations, the last of which is successful (i.e. is not rectified).

The strategy consists of steps A_n, B_n, $n \in \mathbb{N}$. The family $(A_n)_{n \in \mathbb{N}}$ enumerates (F_n). If at some stage we find that (F_n) does not fulfil the purpose of its construction (i.e. $F_n \subseteq W$ occurs for some n) we turn to step B_n (for *that* particular n which witnessed the failure). This F_n-failure step will start a ripple of diagonalizations, succeeding $\Phi^W \neq A; \phi$. Since W is hypersimple, only finitely many A_n will be performed, and at some point a single B_n-step will be activated which (eventually) ends $\mathcal{Q}$'s activity leaving it satisfied. For the diagonalizations we will choose witnesses from $\mathbb{N} - c(\mathbb{N})$ so that we dont interfere with $\mathcal{Q}$. Let $a_1 = 1$, $I_0 = \emptyset$ and assume a constant restraint r from the higher priority requirements. The A_n, B_n steps are as follows:

A_n *(F_n definition)*

1. Define I_n as the set of the next a_n unused (i.e. not in $\cup_{i<n} I_i$) elements in $\mathbb{N} - c(\mathbb{N})$, greater than r and not yet in A. *This is the set of witnesses (agitators) of step A_n. They have the potential to be used by B_n after an A_n-failure. Their number $|I_n| = a_n$ is defined by the previous step A_{n-1}.*
2. Restrain I_n (from A) and wait until $\ell(\Phi^W = A; \phi) > t$, for all $t \in I_n$.
3. Define $F_n := \{\max F_{n-1} + 1, \dots, \max_{i \in I_n} \phi(i) - 1\}$ and $a_{n+1} := |\cup_{i \leq n} F_i| + 1$ $(= \max \phi(\cup_{i \leq n} I_i) + 1)$.

B_n *(F_n-failure diagonalization loop)*

(a) Wait for a Φ-expansionary stage.

(b) Put the least element of $I_n \cap \overline{A}$ into A and go to (a).

The $\mathcal{P}$-module operates as follows: it executes $A_1, A_2, \dots$ but before moving to A_n it checks whether $D_i \cap \overline{W} \neq \emptyset$ for all $i < n$. If this holds, it proceeds to A_n, otherwise it proceeds to B_k for the least k with $D_k \cap \overline{W} = \emptyset$. When the $\mathcal{Q}$ module is called, it starts operating from where it last stopped, until it meets a 'wait' condition which is not fulfilled *or* it finishes an A_n step (in which case it stops at the beginning of A_{n+1}). We start from A_1.

Now that we have defined the operation of $\mathcal{P}$, we explain why this strategy works. First of all note that $D_1, D_2, \dots$ are consecutive segments of $\mathbb{N}$. The outcomes are as follows:

1. when $\mathcal{P}$ executes all A_n. Then, according to the module (F_i) is an infinite disjoint array with $D_i \cap \overline{W} \neq \emptyset$ for all i. This is impossible since W is hypersimple.

2. when we are permanently stuck in a 'wait' instruction in some A_n step. In this case it is obvious that $\Phi^W \neq A; \phi$ and $\mathcal{Q}$ is satisfied.

3. when the above fail, and so the $\mathcal{Q}$-module passes control to some B_n step. This must happen after $D_n \cap \overline{W} = \emptyset$ (i.e. $D_n \subseteq W$) has been noticed by the module.

In the third outcome, B_n will start a ripple of at most $|I_n|$ diagonalizations and we claim that the last one will be impossible to rectify. In other words that $\Phi^W \neq A; \phi$ is a certain final outcome. Indeed, the only rectification codes (i.e. numbers that can rectify Φ^W computations) for any agitator in I_n are in $\mathbb{N} \upharpoonright \max \phi(I_n)$ and so they are not more than $\max \phi(I_n)$. But $\mathbb{N} \upharpoonright \max \phi(I_n) = \cup_{i \leq n} F_i$ and since $F_n \subseteq W$ any rectification code (for witnesses in I_n) is in $\cup_{i<n} F_i = \mathbb{N} \upharpoonright \max \phi(I_{n-1})$. So if K_n is the set of these codes,

$$|K_n| = \max \phi(I_{n-1}) < \max \phi(I_{n-1}) + 1 = a_n = |I_n|.$$

Since for each I_n-enumeration (into A, at a Φ-expansionary stage) at least one K_n-enumeration (into W) is needed for a new expansionary stage to come, there will be a (final) I_n diagonalization which is not rectified. This means that the module will be stuck on (a) of B_n unable to obtain an expansionary stage. So $\Phi^W \neq A; \phi$ and the third outcome satisfies $\mathcal{P}$. Hence the module is successful. Also, note that in each of the two realizable outcomes above the restraints that $\mathcal{P}$ imposes (to lower priority requirements) are finite.

$\mathcal{R}_\phi$ **strategy** Although we were able to find a strategy for $\mathcal{P}$ which does not interfere with $\mathcal{Q}$, it is not possible to do the same with $\mathcal{R}$, since hypersimplicity requirements can not afford to choose their witnesses from a pre-arranged computable set. So we have to allow them to enumerate into elements of $c(\mathbb{N})$ as well and to avoid the destruction of $\mathcal{Q}$ we will take advantage of the hypersimplicity of W once more. Based on the given strong array $(D_{\phi(n)})$ (which tries to show that A is not hypersimple) we will construct a strong array (G_n) which tries to show that W is not hypersimple. When (G_n) fails, i.e. $G_k \subseteq W$ for some k, we will cause a $(D_{\phi(n)})$ -failure (i.e. $D_{\phi(k)}$ for some k) without creating any potential problems to $\mathcal{Q}$. Note that (G_n) will definitly fail since W is given hypersimple. To be more specific, we simply define

$$G_n := \{k \mid c(k) \in D_{\phi(n)}\}.$$

Now since W is hypersimple, some $G_n \subseteq W$ at some stage. But then $\mathcal{R}_\phi$ can be satisfied by putting into A only the elements in $D_{\phi(n)} - c(\mathbb{N})$; indeed, $c(\mathbb{N}) \cap D_{\phi(n)}$ is already in A by $\mathcal{Q}$'s module and $G_n \subseteq W$. In other words we satisfy $\mathcal{R}$ without enumerating into A any c-codes (such an enumeration is left to $\mathcal{Q}$).

Construction. In order to let all the strategies work together we only need to make sure that lower priority requirements respect the restraint r set by higher ones. Note that only $\mathcal{P}$ impose restraints. Whenever a $\mathcal{P}$ or $\mathcal{R}$ receives attention we initialize all lower priority $\mathcal{P}$-requirements. Every $\mathcal{P}$ chooses witnesses greater than the restraint r and restrains them; $\mathcal{R}$ only enumerates into A a G_n with all members greater than r. A $\mathcal{P}$ requirement requires attention when its module is ready to move to the next step; and a $\mathcal{P}$ requirement when there is a G_n with $G_n \subseteq W$ and $\min G_n > r$. The *construction* is: at stage s

- For every n, if $n \in W$ (and $c(n) \notin A$) put $c(n) \searrow A$.
- Find the least $\mathcal{P}$ or $\mathcal{R}$ which requires attention in the first case run the relevant module (from where it last stopped) and in the latter find the least n with $G_n \subseteq W$, $\min G_n > r$ and enumerate the elements of $D_{\phi(n)}$ into A. Initialize the lower priority requirements.

The satisfaction of the requirements follows by the analysis of outcomes we discussed above and an application of the finite injury method. □

6.5 Hypersimple Semicomputable Sets in the Wtt degrees

In the previous sections we dealt with the notion of hypersimplicity and now we consider how semi-computability (in the sense of Jockusch[19]) relates to the wtt c.e. degrees along with hypersimplicity. We recall the following definition:

Definition 30 (Jockusch[19]). *A set A is semicomputable if there is a computable f such that*

- $f(x, y) \in \{x, y\}$
- $x \in A \ \vee \ y \in A \Rightarrow f(x, y) \in A$.

Semicomputable sets are known to be exactly the cuts of computable linear orderings of $\mathbb{N}$ and as Jockusch[19] points out,

Proposition 12 (Jockusch[19]). *Every c.e. wtt (and indeed tt) degree contains a c.e. semicomputable set.*

So it makes sense to study the wtt degrees of sets that are both hypersimple and semicomputable. First we provide a characterisation of the hypersimple semicomputable sets, which will give a better intuition in our constructions.

Theorem 25. *A set is hypersimple semicomputable iff it is the left c.e. non-computable cut of a computable ordering of* $\mathbb{N}$ *of type* $\omega + \omega^*$.

Proof. It will be clear that 'left' can be replaced by 'right'. As mentioned above, it is well known that semicomputable sets are exactly the cuts of computable orderings of $\mathbb{N}$. Also, it is not difficult to show that if a cut of a computable ordering of $\mathbb{N}$ of type $\omega + \omega^*$ is c.e. non-computable, then it is hypersimple (see chapter 3[1]). Hence one direction of the theorem follows easily.

For the other, assume that A is semicomputable and hypersimple. Then it is the left cut of a computable ordering $\prec$ of $\mathbb{N}$. Assume an effective enumeration A_s of A (with $\max A_s < s$) and define the set B as follows:

stage s. If s lies on the $\prec$-left of some element in A_s, enumerate $s \searrow B$.

Obviously B is a computable subset of A. It is the set of elements which we know they belong to A, by the time they are enumerated in the standard enumeration of $\mathbb{N}$. We will define a new order $\prec_*$ of $\mathbb{N}$ which is of type $\omega + \omega^*$ and its left cut is A. In fact, $\prec$ and $\prec_*$ will only differ on B.

The intuition is that in order to transform the order type of $\prec$ to $\omega+\omega^*$ it is sufficient (and necessary) to ensure that every element has either finitely many predecessors or finitely many successors. Since A is infinite, any element of $\overline{A}$ has infinitely many $\prec$-predecessors and so we must ensure that it has only finitely many $\prec_*$-successors. Similarly, for the elements in A we must ensure that they have only finitely many predecessors, and we do this by reordering some of them.

We view the construction of $\prec_*$ as mapping (placing) natural numbers into a dense line like $\mathbb{Q}$. The order of the rationals induces $\prec_*$ via the mapping. In fact, we already have such a mapping with respect to $\prec$. Thus we only have to *move* some naturals on the line, and this re-placement will define $\prec_*$. At stage s it is enough to specify the position of s with respect to the numbers in $\mathbb{N} \upharpoonright s$. Here is the construction. Run the construction of B as above and at stage s, if $s \searrow B$ we place s between the two $\prec$-largest elements in $A_s - B$ (and larger than every B-element currently in there). If not, we leave it in its old position.

Note that $\prec$ is a (computable) order; also, we only move elements in the left cut A of $\prec$ and the new positions remain in A. So A is a left cut of $\prec_*$ as well. Now if there

[1]in terms of that chapter, theorem 25 can be restated as 'semicomputable hypersimple sets are exactly the approximation representations (of c.e. reals)'.

was an element in $\overline{A}$ with infinitely many $\prec_*$-successors, there would be an infinite c.e. subset of $\overline{A}$. This contradicts the hypersimplicity assumption. The only thing left to show is that any element of A has finitely many $\prec_*$-predecessors. Indeed, by induction every element in A has a $\prec_*$-successor in A. If $t \in A$ and $t \prec_* k$, every $s > \max\{t, k\}$ will be $\prec_*$-greater than t. This concludes the proof. □

What we did in the above construction is to spot elements which are 'born' too low (i.e. too left on the line) and lift them as much as possible within the least initial segment of the line which contain all elements of A (we sometimes call this 'black area'). The set B contains the 'low-born' elements (of the black area). Theorem 25 shows that the *approximation representations* (or simply *representations*) for c.e. reals studied in chapter 3 are just the subsets of $\mathbb{N}$ that are both hypersimple and semicomputable. Below we will often say 'representation' instead of 'hypersimple semicomputable set'. Also, a representation or hypersimple semicomputable wtt degree is one that contains representations. The reason why we use this term is that technically speaking we do not see these sets as a combination of the two classical notions but rather as sets with an easily identifiable and intuitively clear structure (as cuts of computable orderings of a special type). The building of a representation will be a construction of a computable linear ordering of type $\omega + \omega^*$ with a simultaneous infinite enumeration of its left cut (roughly as in the proof of theorem 25).

6.5.1 No greatest element for the hypersimple semicomputable wtt degrees

It is natural to ask whether there is a 'universal' c.e. cut of a $\omega+\omega^*$ computable ordering of $\mathbb{N}$ in the sense that it $\leq_{\mathrm{wtt}}$-bounds every other of the same kind. We give a negative answer by showing that there is no maximum hypersimple semicomputable wtt degree, i.e. one that bounds all the others.

Theorem 26. *There is no greatest hypersimple semicomputable wtt degree.*

For the proof we assume we are given a representation A and we construct a representation B such that $B \not\leq_{\mathrm{wtt}} A$. We want to satisfy the following:

$$\mathcal{Q}_\Phi : \Phi^A \neq B; \phi$$

Here Φ runs over the partial computable functionals and ϕ over the partial computable functions (intended as the use of Φ). The plan is to diagonalize against

$\Phi^A = B; \phi$ in a way that is impossible (for the opponent) to rectify (by A-enumeration). For this we will need to diagonalize a number of times, of which the first one (with witness b in the module below) has a special role. We choose b along with a finite number of other witnesses so that after the rectification of $b \searrow B$ the A-enumeration triggered (the set D in the module below) has left less rectification points with respect to our other witnesses than the number of these very witnesses. This guarantees that when we start successive diagonalizations with the other witnesses (at expansionary stages) at least one of them will be impossible to rectify. For this plan the fact that A is a representation is crucial.

We view A as the left bi-infinite cut of a computable $\omega + \omega^*$ ordering $<_A$ of $\mathbb{N}$; so we are given $<_A$ and an enumeration of A. The construction will define a computable ordering $<_B$ of $\mathbb{N}$ of the same type and simultaneously enumerate its unique left bi-infinite cut in B. We view the definition of $<_A, <_B$ as taking place on an A-line and B-line respectively (since they are linear). The enumeration of a cut is represented graphically by a c.e. *black area* (see figure 6.2) which is continuously expanding and eventually covers the part of the line that contains elements in the cut. The elements of $\mathbb{N}$ are also called points when they are mentioned in relation to the A or B-line. Another way to say $m <_A n$ for two numbers m, n is that m is on the left of n (or n on the right of m) on the A-line (see figure 6.2). At any stage only a finite segment of $\mathbb{N}$ is $<_A$ (or $<_B$) -ordered and so, as we say, the numbers in this segment have a position on the corresponding line.

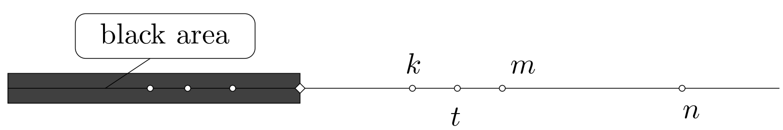

Figure 6.2: Construction of a c.e. cut of a $\omega + \omega^*$ computable ordering.

In the module below we use the symbols ∞_A, ∞_B which refer to the A and B lines respectively. These have the properties $n >_A \infty_A$, $n \not\leq_A \infty_A$ for all n, $\infty_A \notin A$ and similarly for B. Intuitively they are a non-standard point on the corresponding line, on the left of any standard one and we use them just to make our description simpler and more uniform. To save space, we talk about the strategy as if there is potential for $\mathcal{Q}$ to work with other similar requirements. However it can also be seen as the module of $\mathcal{Q}$ working in isolation. We use the *origins* o_A, o_B (as parameters of $\mathcal{Q}$) on the A, B lines respectively which are the points defining the segments on these lines involved in higher priority requirements' activity (if any). So $\mathcal{Q}$ uses points on the left of the origins and it also assumes that $o_A \notin A$, i.e. that no point on the A-line related to a higher requirement enters A. If this assumption is false it will be initialized.

$\mathcal{Q}_\Phi$-module

1. Define the origin o_B of this requirement on the B-line as the leftmost point currently outside B (if such doesn't exist, let $o_B = \infty_B$). Put the next number b without a position on the B-line, to the left of o_B. Set $I_B = \{b\}$.

   Define the origin o_A of this requirement on the A-line as the leftmost point in the I_A-intervals of higher requirements (if they don't exist or they are empty, let $o_A = \infty_A$). Dynamically define the set I_A for this requirement as:

   $$I_A = \{i \mid i < \max_{k \in I_B} \phi(k) \wedge i <_A o_A\} - D - A$$

   where D is dynamically defined $= \{i \mid i \leq_A \min_A\{t < \phi(b) \mid t \notin A\}\}$; $\min_A$ is the $<_A$-minimum and $\min_A \emptyset = \infty_A$.

   The dynamic definition of a parameter means that whenever mentioned it is (re-) defined by applying the definition using the current values of any parameters involved. The set I_B contains the agitators that we plan to use for our diagonalizations. D is the set of numbers that will enter A if the diagonalization $b \searrow B$ of the next step is rectified. So I_A is the set of elements that can rectify I_B-diagonalizations after $b \searrow B$ has been rectified.

2. Wait until $\ell(\Phi^A = B; \phi)$ is greater than all elements of I_B and ask: is $|I_B| > |I_A|$?

   - Yes: Put $b \searrow B$ and redefine dynamically $I_A := I_A - A$ (the right hand side I_A having the value it was last assigned); go to step 4.
   - No: go to step 3.

   If the 'yes' clause holds, then we can start the diagonalization ripple of step 5 and I_A indeed contains the only rectification codes we have to deal with. After $b \searrow B$, D plays no role in the definition of I_A and so we fix the latter. The redefinition $I_A = I_A - A$ is just a way to express that whenever a point of I_A enters A, then it exits I_A (not being a rectification point anymore).

3. Put the least number m not having a position on the B-line, on the right of b and on the left of any other point currently on the line and $>_B b$ (where $>_B$ is the ordering of $\mathbb{N}$ associated with the representation B we are constructing). Define

$$I_B := I_B \cup \{m\}$$

and go to step 2.

4. *(diagonalization loop)*

   (a) Set $I_B = I_B - B$ and wait until the next expansionary stage.

   (b) Put the leftmost point of I_B into B (and expand the black area up to that point) and go to (a).

Note that in contrast to I_A's dynamic definition, I_B has the value it was last assigned. I_A is not necessarily an interval in the A-line (in the sense of the set of points contained in between two points) but it is an interval restricted to numbers in an initial segment of $\mathbb{N}$. However I_B is an interval (on the B-line).

Analysis of Outcomes

By its definition, I_A will only be finite in the long run (due to the fact that A is given as a representation); and since we keep putting elements into I_B, at some point (having enough expansionary stages) we will exit step 2 through the 'yes' clause. After step 4(a) every rectification point for any of our I_B witnesses will be in I_A. And since $|I_B| > |I_A|$ (which will always hold since a I_B-diagonalization can only be rectified by a I_A-enumeration) the loop of step 4 will terminate on (a) due to the lack of an expansionary stage. So $\mathcal{Q}$ is satisfied.

Construction

For every requirement, if its origin o_A has entered A, initialize all higher priority requirements (i.e. initialize their module and enumerate their I_B set into B). Otherwise consider the highest priority $\mathcal{Q}$ requiring attention (i.e. ready to perform the next step) and activate it; initialize all the lower priority requirements.

Verification

We prove by induction that every $\mathcal{Q}$ ceases to require attention and is satisfied with I_A, I_B ending up either undefined (if $\Phi; \phi$ partial) or fixed finite sets. Suppose that all higher priority requirements than $\mathcal{Q}$ are satisfied in this way (and beyond a least stage

s_0 they don't require attention). This means that o_A of $\mathcal{Q}$ is eventually settled on a final value outside A.

For a contradiction suppose that $\Phi^A = B; \phi$. Step 1 will run and we claim that the loop of steps 2-3-2 will produce what we call a *saturation state* i.e. a stage that $|I_B| > |I_A|$. Indeed, consider the final value of $t = \min_A\{t < \phi(b) \mid t \notin A\}$. If it is ∞_A then $I_A = \emptyset$ and the inequality holds. Otherwise, only finitely many points can be in $[t, \infty_A)$ since A is representation. So I_A is finite and by successively adding elements in I_B we will eventually get $|I_B| > |I_A|$.

From now on every diagonalization using elements of I_B requires A-enumeration of elements in I_A; indeed, it holds for the diagonalization with b, and any other point below the use of some I_B computation either belongs to D (as it was defined before executing the 'yes' clause of step 3) or I_A, or it is $\geq_A o_A$. But by the time the b-diagonalization is rectified, $D \subset A$ and by hypothesis no element $\geq_A o_A$ is ever going to enter A. This means that any subsequent rectification must be done with elements in I_A. Now that we found a suitable I_A, we fix it except for the fact that we update it by deleting points that have entered A (and so, are useless for rectification).

The (a)-(b) loop of step 4 will keep on reducing I_B ensuring that for each I_B enumeration at least one I_A-enumeration happens. And elements in I_B can only be enumerated by $\mathcal{Q}$ (at expansionary stages) due to the hypothesis (the rest of higher priority requirements) and the fact that lower priority requirements choose I_B-witnesses on the left of those of higher ones. So at some point $|I_A| = 0$ while $I_B \neq \emptyset$ and a further diagonalization with an I_B witness will be impossible to rectify. So $\Phi^A \neq B; \phi$, a contradiction. Since $\Phi^A \neq B; \phi$, after a certain stage there will be no more expansionary stages. So I_B will stabilize and I_A as well (by its definition).

6.5.2 Hypersimple Semicomputable wtt degrees and the join

Since we are studying the class of hypersimple semicomputable wtt-degrees it is natural to ask whether they are closed under join. We show that they are not; moreover, we construct two hypersimple semicomputable sets such that any set which can wtt-compute both of them, is not hypersimple semicomputable.

Theorem 27. *There are hypersimple semicomputable A,B such that no $W \geq_{wtt} A \oplus B$ is hypersimple semicomputable.*

Let $A \oplus B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}$ where $\langle .,.\rangle$ is a standard pairing function. We want to satisfy the following:

$$\mathcal{Q}_{\Phi,W,\theta} : W \text{ is representation via } \theta \ \Rightarrow \Phi^W \neq A \oplus B; \phi$$

Here Φ runs over the partial computable functionals, W over the c.e. sets and θ over the partial computable functions. The phrase 'W is a representation via θ' means that W is the left cut of the computable ordering of $\mathbb{N}$ determined by θ (in the sense that $n \prec_\theta m \iff \theta(n,m) = 1$) and this ordering has type $\omega + \omega^*$ (ω^* is the inverse of ω). Here we use the fact that representations are exactly the left cuts of such orderings in order to test this property over the list of c.e. sets. In the following when we talk about a particular requirement, θ will only be implicit; i.e. we will talk about a point (i.e number) t being 'on the left' of another k (on the W-line) meaning that $t \prec_\theta k$ (and analogously for 'on the right').

Of course if we only had one representation instead of A, B above, the satisfaction of the requirements would be impossible (and it is instructive to see why). The problems in that situation can be solved if we share our diagonalization witnesses between two sets. The strategy is to gather enough suitable witnesses so that if W is indeed a representation (via θ) and we put each witness into $A \oplus B$ in successive (Φ-) expansionary stages, the W-enumeration we will cause (needed for rectification of Φ^W) is enough to guarantee impossibility of rectification by the time we enumerate the last witness. If W is a representation, we can trigger massive enumerations into W with just one diagonalization since if a point enters W, all points on its left enter W as well; and if $t \notin W$ almost all points are on the left of t. For this plan, the first of our witnesses is the one which triggers a massive W-enumeration and the others just need a usual W-enumeration (i.e. one element below the use). Since we definitely want to diagonalize with a particular witness t before all the others and the sets we are building must be representations, we should either

1. enumerate t in one of A, B and the rest in the other; or
2. all in the same set but in this case t must be on the left of all the other witnesses (because otherwise its enumeration will cause other witnesses to be enumerated as well, before they are used).

If W is not a representation (a fact that we cannot predict effectively) the above plan does not work, simply because we may not find suitable witnesses, able to trigger desired W-enumerations (but $\mathcal{Q}$ is satisfied in a trivial way). However, this situation may induce an infinite search for witnesses, and if we choose to act as in (2) we may destroy the representation structure of A or B. So we choose to follow (1) and this is why we need to use diagonalization witnesses from two sets (A and B) instead of one.

We use A for our initial witness and B for the rest ones. In this situation we do restrain our A-witnesses but we don't restrain the B ones unless we are sure we have got enough (to start the diagonalization ripple). So the 'infinite search' described above will have no significant effect in the construction (e.g. in terms of restraints). This approach assumes that B is co-infinite (so that we are able to find arbitrarily many potential witnesses) before we are able to show the satisfaction of the requirements. This assumption is justified (i.e. can be proved) by allowing $\mathcal{Q}_n$ to use B-witnesses only beyond (in particular, to the left of) a certain point $p(n)$—the n-th point outside B counting from right to left—which takes a final value in the course of the construction.

Its time to turn this informal discussion into a formal strategy for a single requirement, the $\mathcal{Q}_{\Phi,W,\theta}$ module described below. To save space, we present it as the module of $\mathcal{Q}_n$ (assuming that $\mathcal{Q}_{\Phi,W,\theta}$ is the n-th requirement in an effective list $\mathcal{Q}_0, \mathcal{Q}_1, \ldots$ of all requirements); this does not affect the clarity of the presentation since we can easily get the atomic module (i.e. $\mathcal{Q}_{\Phi,W,\theta}$ working in isolation) by fixing n and considering r (the restraint imposed by higher priority requirements) to be 0. The length of agreement of $\Phi^W = A \oplus B; \phi$ is $\ell(\Phi^W = A \oplus B; \phi)$. By convention we assume that $\Phi^W(t)\downarrow$ implies that all the numbers below the use of the computation have been ordered by θ.

Recall the intuition we built in the proof of theorem 26 on constructing a representation: here we also have A and B lines and a black area for each of these (see figure 6.2). The current value of $\overline{B}$ is the set of elements having been assigned a position on the B-line and being currently outside B. At each stage s the construction (stated later) will order s on the left of any point outside A on the A-line, and similarly for B. This can be seen as building the orderings of $\mathbb{N}$ associated with the representations A, B. To be consistent with their representation nature, whenever an action enumerates a point into A or B, we assume that all points on its left are also enumerated into the same set (in our terminology, we expand the black area of the corresponding set up to that point).

$\mathcal{Q}_{\Phi,W,\theta}$-module

1. Choose an A-agitator $a \in \overline{A}$ on the left of any (current) A-agitator of a higher requirement.

2. Wait until $\ell(\Phi^W = A \oplus B; \phi) > \langle 0, a \rangle$.

3. Wait until

    (a) $|\overline{B} - R| > |E|$

    (b) $\ell(\Phi^W; A \oplus B) > \langle 1, b\rangle$ for all $b \in I$

    where

    - $p(n)$ is the n-th point (from right to left) on the B-line, outside B.
    - $R = \{t \in \overline{B} \mid t \geq_\theta p(n)$ or $\exists k \leq r(t \geq_\theta k \wedge k \notin B)\}$. *These are the restrained points.*
    - $E = \{t \mid t >_\theta \min_\theta(\overline{W} \upharpoonright \phi(\langle o, a\rangle))\}$ *(it includes the rectification codes against our planned diagonalizations, at any stage after $a \searrow A$);* $\min_\theta$ is the minimum with respect to θ and by convention $\min_\theta \emptyset = \infty_\theta$, a symbol with the properties $\infty_\theta \notin W$ and for all n, $n <_\theta \infty_\theta$ and $\infty_\theta \not\leq_\theta n$.
    - I is the set of the first $|E| + 1$ points on the B-line outside B and after (i.e. on the right of) any element of R. *It is the set of B-witnesses for our future diagonalizations and is defined provided that the first condition is satisfied.*

Note that if there are less than n elements on the B-line outside B, $p(n)$ is undefined. R is the set of restrained elements; the component r comes from the higher priority requirements and the component $p(n)$ comes from our intention to make sure that $\overline{B}$ is eventually infinite. E contains the codes that can rectify the B-motivated diagonalizations we plan to do (for which we are searching witnesses in this step) except the ones which are on the left of the leftmost rectification code for $\Phi^W \neq A \oplus B; \phi$ on $\langle 0, a\rangle$ that will be created on step 4. These additional codes will vanish after step 4 and so we need not take them into account. The symbol ∞_θ is analogous to ∞_A or ∞_B that we used in the proof of theorem 26.

The first condition asks for a number of points on the B-line outside B and outside the restrained segment R, greater than the number of elements which can rectify the diagonalizations that can be performed using the former as witnesses. If it is satisfied, we are guaranteed a successful diagonalization. Conversely, if indeed W is a representation via θ, E will be (eventually) finite and since $\overline{B}$ is infinite the condition will be satisfied. Finally, the second condition, if satisfied, makes sure that all rectification codes for our potential diagonalizations have been taken into account in E. Note that every parameter has a current value; e.g. E considers only points (numbers t)) that are currently defined on the W-line.

4. Restrain I and put $a \searrow A$. Dynamically redefine $E = E - W$. *Once we find suitable B-witnesses we restrain them from B for later use. Note that this restraint is for the lower priority requirements, not $\mathcal{Q}_{\Phi,W,\theta}$ itself (or the higher ones). The enumeration of our A-witness into A triggers the ripple of diagonalizations that are going to follow (as long as we get Φ-expansionary stages). It makes sure that after the next expansionary stage E (as it was defined just before we enter this step) will indeed contain every possible rectification code (and so the plan is sound). Moreover we fix E to its last value (which is what we were looking for), with the exception that elements that enter W are deleted form E as they have no rectification potential; this way, at any time after this step, E will indeed be the set of rectification codes against our diagonalizations.*

5. *(diagonalization loop)*

    (a) Wait until the next expansionary stage.

    (b) Put the leftmost point of $I \cap \overline{B}$ into B (and expand the black area up to that point) and go to (a).

Analysis of Outcomes

Requirement $\mathcal{Q}$ works on the assumption that the higher requirements have ceased to require attention (i.e. have rested). If this is false, it will be initialized. From the module described above it follows that every requirement eventually rests (since there are no infinite loops—I is finite) and so in this analysis of outcomes we can assume that all higher requirements have rested (or that we work with a single requirement in isolation).

If we don't have the chance to perform step 1 it will be because of the lack of expansionary stages and so $\mathcal{Q}$ is satisfied in a very trivial way. The rest of the outcomes are listed below:

$\boxed{w_1}$: we wait in step 2 forever. Then $\Phi^W;\phi$ is partial and $\mathcal{Q}$ is satisfied.

$\boxed{w_2}$: we wait in step 3 forever. Then either we cease getting expansionary stages ($\mathcal{Q}$ satisfied) or each time we get them one of the conditions in step 3 fails. Since $\overline{B}$ is infinite (this is a working hypothesis which will be the first thing to prove in the verification and it does not depend on this analysis),

$$|\overline{B} - R| \to \infty \text{ as } s \to \infty$$

and so $|E| \to \infty$ as $s \to \infty$. But this means that $\min_\theta(\overline{W} \restriction \phi(\langle 0, a\rangle))$ is a point on the W-line (and not ∞_θ) and so W is not a representation. Hence $\mathcal{Q}$ is satisfied and no B-restraints are imposed.

$\boxed{w_3}$: we wait in step 5(a) forever. Again, $\Phi^W; \phi$ partial and $\mathcal{Q}$ is satisfied.

Finally there is a possibility that we are in 5(b) and unable to execute it because $I \cap \overline{B} = \emptyset$. We show that this cannot happen; indeed when we leave step 4 we hold (in I) $|E| + 1$ elements of $\overline{B}$ and these will not enter B unless $\mathcal{Q}$ instructs so (since they are restrained). An enumeration of any of them at an expansionary stage will require W-rectification.

Before leaving step 4 we also put $a \searrow A$ currently being at an expansionary stage, which means that before running step 5 some $t \in \overline{W} \restriction \phi(\langle 0, a\rangle)$ must enter W (for the diagonalization to be rectified). After this W enumeration any point that can rectify an I-diagonalization is in E: indeed, it had a position on the W-line when we left step 3 and at that time it was $>_\theta \min_\theta(\overline{W} \restriction \phi(\langle 0, a\rangle))$ (otherwise it would have entered W by now). Now every time we return to 5(a), $|E \cap \overline{W}|$ will be (at least) one less than it was before; and since $|I| = |E| + 1$ (here E is as it was defined when we left step 3) when we spend our last I-diagonalization, $E \cap \overline{W} = \emptyset$ already and a rectification (and so, leaving (a)) will be impossible.

Construction

At stage s put s on the A, B lines (outside the black area) on the left of any existing point outside the black area. Consider the least $\mathcal{Q}$ requiring attention (i.e. ready to perform the next step) and run the corresponding module. Initialize all lower priority $\mathcal{Q}$ requirements.

Verification

First we verify our working hypothesis.

Lemma 27. *$\overline{B}$ is infinite.*

Proof. Suppose not, i.e. that $p(n) \to \infty$ as $s \to \infty$ for a least n. By the $\mathcal{Q}$ module, no Q_i, $i \geq n$ can act enumerating (some value of) $p(n) \searrow B$. And since $p(n)$ is (enumerated and) redefined infinitely often, there must be a least $\mathcal{Q}_i, i < n$ which enumerates values of $p(n)$ into B infinitely often. But this is not possible since each $\mathcal{Q}$ only requires attention finitely often (given the finitary nature of the module—there are no infinite loops since I is finite). □

And now, by an adaptation of the analysis of outcomes discussed earlier we can show that each $\mathcal{Q}$ is satisfied. Suppose that $\mathcal{Q}_i, i < n$ have stopped requiring attention. After the last time they received attention, $\mathcal{Q}_n$ will start anew. If it does not execute step 3 (and so 4) its satisfaction follows as in the analysis of outcomes. Otherwise steps 3,4 run and any rectification point on the W line (at any stage) is either in E (as it was defined when step 3 run) or on the left of the leftmost point $< \phi(\langle 0, a \rangle)$ on the W-line outside W. So, since $|I| > |E|$ the loop in step 5 has to stop at some point due to the lack of expansionary stages, thus satisfying $\mathcal{Q}_n$.

Bibliography

[1] K. Ambos-Spies and A. Kučera, Randomness in Computability Theory, in *Computability Theory and its Applications*, Contemporary Mathematics Volume 257, 2000.

[2] G. Barmpalias, A transfinite hierarchy of reals, *Mathematical Logic Quarterly*, Volume **49**, Number 2 (2003) 163-172.

[3] G. Barmpalias, On $0'$-computable reals, *Electronic Notes in Theoretical Computer Science* **66** Issue 1, (2002).

[4] G. Barmpalias, The approximation structure of a computably approximable real, *Journal of Symbolic Logic* **68** no.3 (2003) 885–922.

[5] G. Barmpalias, Approximation Representations for Δ_2 Reals, *Archive for Mathematical Logic*, Volume **43**, Number 8, (2004) p. 947–964

[6] G. Barmpalias, Approximation Representations for Reals and their wtt-Degrees, *Mathematical Logic Quarterly* Vol. 50 number 4/5 (2004).

[7] G. Barmpalias, Hypersimplicity and Semicomputability in the Weak Truth Table Degrees, *submitted*.

[8] J. Blank, V. Brattka, P. Hertling, *Computability and complexity in analysis: CCA 4th International workshop, Swansea, UK, Sept. 2000*, Springer 2001.

[9] C. S. Calude, R. Coles, P. H. Hertling, and B. Khoussainov. Degree theoretic aspects of computably enumerable reals. In S. B. Cooper and J. K. Truss, editors, *Models and Computability*, volume 259 of *London Math. Society Lecture Note Series*, pages 23–39, Cambridge, 1999. Cambridge University Press. Invited Papers from Logic Colloquium 1997, Leeds.

[10] C. Calude and P. Hertling, Computable approximations of reals: An information-theoretic analysis, *Fundamenta Informaticae* **33** (1998) 105–120.

[11] S. B. Cooper, Minimal pairs and high recursively enumerable degrees, *J. Symbolic Logic* **39** (1974), 655–660

[12] R. G. Downey, *Some computability-theoretical aspects of reals and Randomness*, Notes based upon a short course of lectures given in the fall of 2000 at the University of Notre Dame, September 10, 2001.

[13] R. G. Downey, D. R. Hirschfeldt and G. LaForte Randomness and reducibility, in MFCS'01, Lecture Notes in Computer Science **2136** Springer-Verlag, Berlin 2001, 316–327.

[14] R. G. Downey, D. R. Hirschfeldt and André Nies Randomness, computability, and density *SIAM Journal of Computation*, Vol. **31** No. 4, pp. 1169–1183 (2002)

[15] R. G. Downey, G. LaForte Presentations of computably enumerable reals *Theoretical Computer Science*, Vol. **284** (2002), no. 2, 539–555.

[16] A.J. Dunlop and M.B. Pour-El, The degree of unsolvability of a real number, in [8] 16–29.

[17] S. Fenner and M. Schaefer, Bounded immunity and btt-reductions, *Mathematical Logic Quarterly*, Volume **45** (1999), pages 3-21.

[18] C.-K. Ho, Relatively recursive reals and real functions, *Theoretical Computer Science*, **210** (1999) 99–120.

[19] C. Jockusch, Semirecursive sets and positive reducibility, *Transactions of the American Mathematical Society* **131** (1968).

[20] Leonhardi, Steven D. Nonbounding and Slaman triples, *Ann. Pure Appl. Logic* **79** (1996), no. 2, 139–163

[21] M. Li and P. Vitaniy. *An introduction to Kolmogorov Complexity and its Applications*, 2nd ed., Springer-Verlag, New York, 1993

[22] D. Martin Measure, Category, and Degrees of Unsolvability. Unpublished manuscript, dating from the late 60's.

[23] P. Odifreddi, *Classical recursion theory*, Amsterdam Oxford: North-Holand, 1989.

[24] P. Odifreddi, *Classical recursion theory Volume II*, Amsterdam Oxford: North-Holand, 1999.

[25] Pour-El, Marian B. and Richards, J. Ian, *Computability in Analysis and Physics*, 1989, Perspectives in Mathematical Logic, Springer, Berlin

[26] R. Rettinger and X. Zheng, Hierarchy of the monotonically computable real numbers, MFCS'01 August 27-31, 2001, Marianske Lazne, Czech Republic, 633–644.

[27] R. Rettinger, X. Zheng, R. Gengler and B. von Braunmhl, Monotonically computable real numbers, *Mathematical Logic Quarterly*, **48** no 3 (2002) 459–479.

[28] H.J. Rogers, *Theory of recursive functions and effective computability,* McGraw-Hill, 1967.

[29] Shore, Richard A.; Slaman, Theodore A. Working below a high recursively enumerable degree, *J. Symbolic Logic* **58** (1993), no.3, 824–859

[30] R. I. Soare, Computational Complexity, Speedable and Levelable Sets, *Journal of Symbolic Logic* **41** (1977) 545–563.

[31] R. I. Soare, *Recursively enumerable sets and degrees*, Berlin London: Springer-Verlag, 1987.

[32] K. Weihrauch and X. Zheng, A finite hierarchy of recursively enumerable real numbers. MFCS'98, Brno, Czech Republic. Aug. 24 - 28, 1998, LNCS **1450** 798–806, Springer 1998.

[33] K. Weihrauch and X. Zheng, *The Arithmetical hierarchy of real numbers, Mathematical Logic Quarterly*, **47** no.1 (2001) 51–65.

[34] X. Zheng and G. Barmpalias, On the Monotonic Computability of Semi-computable Real Numbers, *Lecture Notes in Computer Science*, Springer-Verlag, Heidelberg, Volume 2731 (2003) 290-300

[35] X. Zheng, R. Rettinger, G. Barmpalias, h-Monotonically Computable Real Numbers, to appear in *Mathematical Logic Quarterly.*

[36] X. Zheng, Recursive approximability of real numbers, *Mathematical Logic Quarterly*, **48** supp. 1 (2002) 131–156.

[37] X. Zheng, K. Weihrauch and K. Ambos-Spies, Weakly computable real numbers, *Journal of Complexity* **16** (2000) 676–690.